DISCRETE-TIME BULK SERVICE QUEUING PROCESSES

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ABSTRACT

In this paper an analysis of bulk service queuing system of Bailey and Jaiswal is made considering time as a discrete variable. Expressions for Average and Variance of queue length and average waiting-time are obtained when the input distribution is binomial and service-time is identically and independently distributed. The results of Bailey and Torben Meisling are shown as particular cases. These results are applied to find the Mean and Variance of queue length and mean waiting time in the special cases (*i*) when the service-time is constant and (*ii*) when the service-time follows geometric distribution.

Introduction

This paper deals with bulk service queuing system for which time is treated as a discrete variable. Mean queue length and mean waiting time are derived for the general case. As is expected, the results of Bailey[1] for continuous-time system are obtained by a suitable limiting process. Also the expressions for the generating function of queue length probabilities and average queue length of Torben Meisling[2] are shown as a particular case, when the units are served singly.

In a discrete-time system, the events can occur only at definite time points called "time marks". Let the time marks be regularly spaced with interval $\triangle t$. Illustrations of such systems are found in electronic installations with internal clocks which control all operations within the system.

The Queue System

The system under consideration consists of a service facility which takes for service a batch of fixed number of s customers or the whole queue length, whichever is less. The units arrive at random at time marks and form single queue in order of arrival. The commencement and completion of service can occur only at time marks. It is also assumed that (*i*) the arrival of a customer at any time mark is independent of the arrival of customer at any other time mark and the probability of arrival is p and no arrival is q such that $p+q=1$; (*ii*) not more than one customer can arrive at a particular time-mark and (*iii*) the service times are identically and independently distributed random variables.

From above it follows that the probability of the number of arrivals m within a time period t_k consisting of k time marks is the binomial probability given by

$$A_{m,k} = \binom{k}{m} p^m q^{k-m} \qquad o \leqslant m \leqslant k \qquad .. \quad (1)$$
$$= o \qquad \text{otherwise}$$

As the expected value of m, $E(m)=kp$, the mean rate of arrival λ is given by

$$\lambda = \frac{k\,p}{k.\triangle t} = \frac{p}{\triangle t} \quad \ldots \quad (2)$$

Let C_k be probability of a service interval v containing k times $\triangle t$ where k is an integer.

$$P_r\,[v = k.\triangle t] = C_k\,,\ (k = o, 1, 2, \ldots\ldots),\ \sum_{k=o}^{\infty} C_k = 1 \quad \ldots \quad (3)$$

$$\text{Then, } E\,(v) = \sum_{k=o}^{\infty} k\ \triangle t.\ C_k = \triangle t.\ \sum_{k=o}^{\infty} k\,C_k \quad \ldots \quad (4)$$

$$E\,[v\,(v - \triangle t\,)] = \sum_{k=o}^{\infty} k\ \triangle t.\ (k-1)\,\triangle t,\ C_k = \triangle t^2 \sum_{k=o}^{\infty} k\,(k-1)\,C_k \quad (5)$$

Let the "traffic intensity" measured in Erlangs be given by

$$\rho = \lambda.\ E\,(v) = p \sum_{k=o}^{\infty} k\ C_k \quad \ldots \quad (6)$$

An assumption $\rho < s$ is made so that the system approaches statistical equilibrium.

It can be noted here that this system approaches to the continuous-time system discussed by Bailey[1] as p and $\triangle t \longrightarrow o$ such $\frac{p}{\triangle t} = \lambda$ a constant, since the binomial input distribution tends to Poisson input distribution.

Technique of Analysis

An imbedded Markov Chain (Kendall[3,4]) has been constructed to characterise the system. For this, the queue length n, used as the state variable, is defined as the number of customers either waiting or being served and the queue length is measured at the beginning of the service interval so that the process becomes markovian. Let the row vector $P=(P_o, P_1, P_2 \ldots)$ contain the steady state probabilities P_i of the system having different queue lengths $i\,(i=0, 1, 2 \ldots\ldots)$. Let $T=(p_{ij})$ be the infinite matrix of transition probabilities. When the system is in the steady state, we must have

$$P = PT$$

In order to find the components p_{ij} of T we introduce the probability b_m that m customers arrive during a service period.

$$b_m = \sum_{k=o}^{\infty} A_{m,\,k}\ C_k = \sum_{k=o}^{\infty} \binom{k}{m} p^m\ q^{k-m}.\ C_k \qquad (m \geqslant o) \quad \ldots \quad (8)$$

$$= o \qquad\qquad (m < c)$$

Let a queue length $i \geqslant s$ at a service epoch become j at the next service epoch. During this service period s customers are serviced from the queue length i and m customers arrive to join the queue.

Therefore $j = i - s + m$, and $b_m = P_{i,\, i-s+m}$

$$\text{i.e. } p_{ij} = b_{j-i+s} \qquad i \geqslant s \quad \ldots (9)$$

When the queue length is $i \leqslant s$ at a service epoch, at the next service epoch the queue length j will be only those who arrive during this service period. Therefore

$$p_{ij} = b_j \qquad i \geqslant s \quad \ldots (10)$$

From this it can be seen that the first $(s+1)$ rows of the transition matrix is the same. So T is given by

$$T = \begin{array}{c|cccccc} \rightarrow & 0 & 1 & 2 & 3 & 4 & \ldots \\ \hline 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ 1 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ 2 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ 3 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ s & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ s+1 & o & b_0 & b_1 & b_2 & b_3 & \ldots \\ s+2 & o & o & b_0 & b_1 & b_2 & \ldots \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

When $s = 1$, T is the same considered by Torben Meisling[2].

Generating Function

From equation (7), P_j is given by

$$P_j = b_j \sum_{i=0}^{s} P_i + b_{j-1} P_{s+1} + b_{j-2} P_{s+2} + \ldots\ldots + b_0 P_{s+j} \quad \ldots (11)$$

Since the generating function of the probabilities P_i's is given by

$g(z) = \sum_{j=0}^{\infty} P_j z^j$, multiplying (11) by z^j and summing over $j=0$ to ∞ and simplifying,

$$g(z) = \frac{\sum_{i=0}^{s-1} P_i (z^s - z^i)}{\left(z^s / h(z) \right) - 1} \qquad \ldots \quad \ldots \quad \ldots (12)$$

Where

$$h(z) = \sum_{n=o}^{\infty} b_n z^n = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \binom{k}{n} p^n q^{k-n} . C_k z^n$$

$$= \sum_{k=o}^{\infty} C_k \sum_{n=o}^{k} \binom{k}{n} (pz)^n q^{k-n} = \sum_{k=o}^{\infty} C_k (pz+q)^k \quad .. \quad (13)$$

The numerator of $g(z)$ in (12) contains s unknown probabilities P_o, P_1, ... P_{s-1}.

Since $g(z)$ $\sum_{n=o}^{\infty} P_n z^n$ is absolutely convergent in $|z| < 1$ it must be regular inside the unit circle $|z| = 1$. Using Rouche's Theorem it can be shown that denominator has $(s-1)$ zeros inside the unit circle. Also there is a zero on $|z| = 1$. As $z=1$ is a root of the numerator, we can determine P_o, P_1, ... P_{s-1} such that the zeros of the numerator and denominator coincide in this region. Let $z_1, z_2, \ldots z_{s-1}$ be the $(s-1)$ zeros of the denominator inside the unit circle. Also let them be simple for convenience.

Since $g(1)=1$ and $h'(1)=\rho$, taking the limit of (12) at $z=1$,

$$\sum_{i=o}^{s-1} P_i \; (s-i) = s - \rho \quad .. \quad .. \quad .. \quad .. \quad (14)$$

Also

$$\sum_{i=o}^{s-1} P_i \left(Z_j^s - Z_j^i \right) = 0 \qquad j = 1, 2, 3, \ldots\ldots (s-1) \quad .. \quad (15)$$

The s equations given by (14) and (15) determine the s unknowns P_o, P_1, ... P_{s-1}. It should be noted that if the denominator of (12) has multiple zeros inside the unit circle, some modification to the above procedure as given by Wishart[5] is necessary.

Since zeros inside the unit circle of the denominator of (12) coincide with those of the numerator, (12) can be written as

$$g(z) = \frac{C \; (z-1) \prod_{j=1}^{s-1} (z - z_j)}{\left(z^s \,/\, h(z) \right) - 1} \quad .. \quad .. \quad .. \quad (16)$$

where $C = \sum_{i=o}^{s-1} P_i$. The constant C can be determined by taking the limit of (16) as z approaches unity from the left and equating the limit to unity, as $g(1)=1$. C is given by

$$C = \frac{s - \rho}{\prod_{j=1}^{s-1} (1 - z_j)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Therefore $g(z)$ can be written as

$$g(z) = \frac{(s - \rho)\ (z - 1) \prod_{j=1}^{s-1} \left(\frac{z - z_j}{1 - z_j} \right)}{\left(z^s / h(z) \right) - 1} \qquad \ldots \quad (18)$$

This expression of $g(z)$ holds good even when all the z_j's are not different. It should be noted when $s=1$, the generating function (18) reduces to that obtained by Torben Meisling[2],

$$g(z) = \frac{(1 - \rho)\ (z - 1)\ h(z)}{z - h(z)} \quad , \quad \rho < 1 \quad \ldots \quad (19)$$

Mean and Variance of Queue Length

Since the mean queue length E(n) can be obtained as the limit of the first derivative of the generating function $g(z)$ as z approaches unity from the left, we have

$$E(n) = \sum_{n=0}^{\infty} n P_n = g'(1) \qquad \ldots \quad \ldots \quad (20)$$

Let the numerator and denominator of $g(z)$ in (18) be denoted by $\phi(z)$ and $\psi(z)$ respectively. As $g(1)=1$ and $\phi(1)$ and $\psi(1)$ vanish at $z=1$, taking the limit of $g(z)$ as $z \geqslant 1$,

$$\phi'(1) = \psi'(1) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Also differentiating $g(z)$ with respect to z and taking the limit as $z \geqslant 1$ using (20)

$$g'(1) = \frac{\phi''(1) - \psi''(1)}{2\,\psi'(1)} = E(n) \qquad \ldots \quad \ldots \quad (22)$$

Differentiating $\phi(z)$ and $\psi(z)$ twice and taking limit as $z \geqslant 1$ using the equations (2), (3), (4), (5), (6) and (13).

$$\phi'(1) = \psi'(1) = s - \rho \qquad \ldots \quad \ldots \quad (23)$$

$$\phi''(1) = 2\,(s - \rho) \sum_{j=1}^{s-1} (1 - z_j)^{-1} \qquad \ldots \quad \ldots \quad (24)$$

and $\psi''(1) = s(s-1) - 2s\rho + 2\rho^2 - \lambda^2 E[\,v(v - \triangle t)]$ $\qquad \ldots \quad \ldots \quad (25)$

Substituting the values of $\psi(1)$, $\phi''(1)$ and $\psi''(1)$ from the equations (23), (24) and (25) in (22) and simplifying,

$$E(n) = \sum_{j=1}^{s-1} (1 - z_j)^{-1} + \rho + \frac{\lambda^2 E[v(v - \triangle t)] - s(s-1)}{2(s-\rho)} \quad (26)$$

similarly, using the formula Var $(n) = g''(1) + g'(1) - [g'(1)]^2$,

$$\text{Var}(n) = -\sum_{j=1}^{s-1} \frac{z_j}{(1-z_j)^2} + \rho(1-\rho) + \lambda^2 E(v(v - \triangle t))$$

$$+ \frac{[\lambda^2 E(v(v-\triangle t)) - s(s-1)][\lambda^2 E(v(v-\triangle t)) - s(s-1) + 2(s-\rho)]}{4(s-\rho)^2}$$

$$+ \frac{\lambda^3 E[v(v-\triangle t)(v - 2\triangle t)] - s(s-1)(s-2)}{3(s-\rho)} \quad (26a)$$

When $s=1$, the first term of the equation (26) vanishes leaving,

$$E(n) = \rho + \frac{\lambda^2 E[v(v - \triangle t)]}{2(1-\rho)}, \; \rho < 1 \quad (27)$$

which is the same as the expression got by Torben Meisling[2] when the units are served singly under the same assumptions.

The result (26) and 26(a) have been applied to find the average and variance of queue length in particular cases, (i) when the service interval is fixed and (ii) when the service interval follows geometric distribution.

Constant service time

When the service time $v=v_o$ a constant,

$$P_r\,[v = v_o = k_o \cdot \triangle t] = C_k = 1 \quad \text{when} \quad k = k_o$$
$$= o \quad \text{otherwise} \quad (28)$$

then $E(v) = v_o$, $E[v(v - \triangle t)] = v_o^2 - \triangle t . v_o$ (29)

$$\text{and} \quad h(z) = (pz+q)^{k_o} = (pz+q)^{\frac{v_o}{\triangle t}} \quad (30)$$

using (30) the generating function of queue length probabilities in (18) reduces to

$$g(z) = \frac{(s-\rho)(z-1) \prod_{j=1}^{s-1} \left(\frac{z - z_j}{1 - z_j} \right)}{z^s [pz+q]^{-\frac{v_o}{\triangle t}} - 1} \quad (31)$$

where z_j $(j = 1, 2 \ldots\ldots s-1)$ are the $(s-1)$ zeros inside the unit circle of the denominator of (31).

The average queue length can now be got either from $g(z)$ in (31) by differentiation and taking limit as $z \geqslant 1$ or directly from (26) using the relation (29).

$$E(n) = \sum_{j=1}^{s-1} (1 - z_j)^{-1} + \rho + \frac{\lambda^2 [v_o^2 - \triangle t.\, v_o] - s(s-1)}{2(s-\rho)} \quad \ldots \quad (32)$$

$$\text{Var}(n) = - \sum_{j=1}^{s-1} \frac{z_j}{(1-z_j)^2} + \rho(1-\rho) + \lambda^2 (v_o^2 - v_o \cdot \triangle t)$$

$$+ \frac{[\lambda^2 (v_o^2 - \triangle t.\, v_o) - s(s-1)]\,[\lambda^2 (v_o^2 - v_o.\, \triangle t) - s(s-1) + 2(s-\rho)]}{4(s-\rho)^2}$$

$$+ \frac{\lambda^3 (v_o^3 - 3v_o^2.\, \triangle t + 2v_o.\, \triangle t^2) - s(s-1)(s-2)}{3(s-\rho)} \quad \ldots \quad \ldots \quad (32a)$$

Noting that $\rho = \lambda \cdot E(v) = \lambda \cdot v_o$ and letting p and $\triangle t \geqslant o$ such that $\frac{p}{\triangle t}$ is a constant λ, (32) and (32a) reduce to

$$E(n) = \sum_{j=1}^{s-1} (1 - z_j)^{-1} + \frac{s - (s-\rho)^2}{2(s-\rho)} \quad \ldots \quad \ldots \quad (33)$$

$$\text{Var}(n) = - \sum_{j=1}^{s-1} z_j (1 - z_j)^{-2} + \frac{s(s+2\rho) + 6\rho(s-\rho)^2 - (s-\rho)^4}{12(s-\rho)^2} \quad (33a)$$

where $z_1, z_2, \ldots\ldots\ldots z_{s-1}$ are the $(s-1)$ zeros within the unit circle of the denominator of the corresponding generating function $g(z)$ in the continuous system. These are the same expressions of Bailey[1] for continuous time system.

Service-time follows Geometric Distribution

Let for $d < 1$, $C_k = d^k (1-d)$, $k = 0, 1, 2 \ldots\ldots$ The generating function in (18) will be

$$g(z) = \frac{(s-\rho)(z-1) \prod_{j=1}^{s-1} \left(\frac{z - z_j}{1 - z_j} \right)}{\frac{1}{(1-d)} z^s [1 - d(pz+q)] - 1} \quad \ldots \quad \ldots \quad (34)$$

since $h(z) = \sum_{k=0}^{\infty} (1-d) d^k (pz+q)^k = \frac{1-d}{1 - d(pz+q)}$ as $| d(pz+q) | < 1$

The denominator of (34) is a polynomial of $(s+1)$ degree in z and as already stated has $(s-1)$ zeros inside the unit circle and also $z=1$ is a root. Let z_s be the

$(s+1)^{th}$ root outside the unit circle $|z|=1$. Therefore $g(z)$ can be written as

$$g(z) = \frac{c_1 (z-1) \prod_{j=1}^{s-1} (z-z_j)}{(z-1) \prod_{j=1}^{s} (z-z_j)} = \frac{c_1}{z-z_s} \quad \ldots \quad (35)$$

Where c_1 is a constant chosen such that limit of $g(z)$ when $z \gg 1$ from the left is equal to unity

Therefore,

$$c_1 = 1 - z_s$$

The generating function $g(z)$ is given by

$$g(z) = \frac{z_s - 1}{z_s - z}$$

which is similar to that of negative exponential service-time distribution got by Bailey[1] since the geometric distribution tends to negative exponential as $\Delta t \to 0$ under suitable conditions.

It can be seen for geometric distribution

$$E[v(v-\Delta t)] = \Delta t^2 . \frac{2d^2}{(1-d)^2} = 2\,[E(v)]^2 \quad \ldots \quad (38)$$

Using (6) and (38) the equation (26) in this case reduces to

$$E(n) = \sum_{j=1}^{s-1} (1-z_j)^{-1} + \frac{s(2\rho - s + 1)}{2(s-\rho)} \quad \ldots \quad (39)$$

where z_j's are the $(s-1)$ roots inside the unit circles $|z|=1$ of the polynomial of $(s+1)$ degrees in the denominator of (34). Since $E(n)$ is also equal to $g^1(1)$, differentiating (37) and putting $z=1$,

$$E(n) = \frac{1}{z_s - 1} \text{ and similarly } \operatorname{Var}(n) = \frac{z_s}{(z_s-1)^2} \quad \ldots \quad (40)$$

where z_s is the one root of the denominator of (34) outside the unit circle.

In the case when $s=1$, the polynomial in the denominator of (34) reduces to quadratic $pdz^2 - z(1-dq) + (1-d)$ whose one root is $z=1$ and the other root is $z = \frac{1-d}{pd}$ which is equal to $\frac{1}{\rho} > 1$. Hence $E(n) = \frac{1}{\frac{1}{\rho} - 1} = \frac{\rho}{1-\rho}$ and

$\operatorname{Var}(n) = \frac{\rho}{(1-\rho)^2}$ These are the results of Torben Meisling[2].

Waiting-Time-Distribution

Since waiting-time distribution should involve the service time distribution and the state probabilities of different queue length, the Laplace Transform of waiting-time distribution $W(a)$ given by Downton [6] and Jaiswal [7] is written as

$$W(a) = \frac{\left(\frac{1}{\beta(\alpha)} - 1\right) \cdot g\left(1 - \frac{1 - e^{-a \cdot \triangle t}}{p}\right)}{E(v) \cdot \alpha} \quad .. \quad (41)$$

where $\beta(a) \sum_{k=o}^{\infty} C_k e^{-a.k\triangle t}$, and $h(z)$ *in* (13) 's $\beta\left[-\frac{\log(pz+q)}{\triangle t}\right]$

The mean and variance of waiting-time ω can be found by successive differentiation of $W(a)$ taking the limit as $a \to 0$ and noting $\frac{p}{\triangle t} = \lambda$.

Here, only the expression for mean waiting-time $E(\omega)$ is given

$$E(\omega) = \frac{E(n)}{\lambda} - E(v) + \frac{E[v(v - \triangle t)] + \triangle t \cdot E(v)}{2E(v)} \quad .. \quad (42)$$

using (2), this reduces to

$$E(\omega) = \frac{1}{\rho}\left[(E(n) - \rho)E(v) + \frac{\lambda}{2}\left\{E\left[v(v - \triangle t)\right] + \triangle t \cdot E(v)\right\}\right] \quad (43)$$

It can be easily seen, by taking the limit as $\triangle t \to 0$ such that $\frac{p}{\triangle t} = \lambda$ a constant, that (43) reduces to the same result as Bailey's[1].

For constant service-time using (29) and (32), the expected waiting-time becomes

$$E(\omega) = \left[E(n) - \frac{\rho}{2}\right]\frac{1}{\lambda} = \left[\sum_{j=1}^{s-1}(1-z_j)^{-1} + \frac{\rho}{2} + \frac{\lambda^2[v_o^2 - \triangle t.v_o] - s(s-1)}{2(s-\rho)}\right]\frac{1}{\lambda}$$

Similarly from (42) using (38) and (39) it can be got when the service-time follows geometric distribution,

$$E(\omega) = \left[\sum_{j=1}^{s-1}(1 - z_j)^{-1} + \frac{s(2\rho - s+1)}{2(s-\rho)} + \frac{\triangle t}{2} \cdot \lambda\right]\frac{1}{\lambda} \quad (45a)$$

$$\text{Also} = \left[\frac{1}{z_s - 1} + \frac{\triangle t}{2}\lambda\right]\frac{1}{\lambda} \quad \text{using (40).}$$

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References

1. Bailey, N. T. J., *J. Roy. Statist. Soc.* B, **16**, 80, 1954.
2. Torben Meisling, *Opns. Res.*, **6**, 96, 1958.
3. Kendall, D. G., *J. Roy. Statist. Soc.* B, **13**, 151, 1951.
4. *Ann. Math. Statist.*, **24**, 338, 1953.
5. Wishart, D. M. G., *Ibid*, **27**, 775, 1956.
6. Downton, F., *J. Roy. Statist. Soc.* B, **17**, 256, 1955.
7. Jaiswal, N. K., *Opns. Res.*, **8**, 139, 1960.