# DISTRIBUTION OF BUSY PERIODS FOR THE BULK-SERVICE QUEUING PROBLEM

by

**N. K. Jaiswal**

**Defence Science Laboratory, Delhi**

## ABSTRACT

The distribution of busy periods for the bulk-service queuing problem, in which the server on finding the system empty, waits, has been obtained by using the 'phase' technique. As particular cases, the results of the classical queue have been derived.

In any queuing process, the server is either occupied with a unit or is idle. The time interval during which the server remains busy is called the busy-period and the distribution of these busy periods is important from the point of view of the server.

The study of busy periods for the simplest queuing process M/M/1 is due to Kendall[1], Ledermann and Reuter[2] and Bailey[3]. For the system M/G/1, the distribution of busy periods is due to Takács[4] whose results were extended by Benes[5] and McMillan and Riordon[6]. Luchak[7] studied busy periods for the case of a queuing process characterised by a time dependent Poisson input and a wide class of service time distribution and later[8] obtained results for the same process under time independent conditions. In both these papers, the 'phase' technique has been employed. Some recent investigations for the queuing processes GI/M/1 and GI/$E_k$ /1 are due to Conolly [9,10] and those for the queuing process M/G/1 are due to Prabhu[11].

In the bulk-service queuing problem studied by Bailey[12], the server is continuously busy and therefore the question of deriving this distribution does not arise. Another type of bulk-service queuing problem, in which the server, on finding the system empty, waits for a unit to arrive, has been studied by the author[13]. This type of process will be called the modified bulk-service process. The various operational parameters of the two processes were studied by the author but the distribution of busy periods for the modified bulk-service process remains yet to be investigated. In this paper, therefore, this distribution is derived and as particular cases, the results of the classical queue have been obtained.

We assume that at $t=0$, the system is empty and a unit arrives which initiates the busy period. The 'phase' technique (Jaiswal[13]) has been used to simulate a wide class of service time distributions. We define $P_{n,r}(t)$ as the probability that at time $t$, there are $n$ units waiting in the queue and the service is in the $r$th phase. Therefore $P_{o,r}(0)=C_r$, because the incoming unit may demand the $r$th phase with probability $C_r$. With this initial condition, we have to determine the distribution of time, the system takes in reaching the empty state for the first time. Therefore if $P_o(t)$ is the probability that at time $t$, the system attains the empty state for the first time, the required probability density is given by

$$\gamma(t) = \frac{dP_o(t)}{dt} \quad \ldots \quad (1)$$

Equations for the process are,

$$\frac{d}{dt}P_{n,r}(t) = -(\lambda+\mu)P_{n,r}(t) + \lambda P_{n-1,r}(t) + \mu P_{n,r+1}(t) + \mu C r P_{n+S,1}(t) \quad (n > 0, 1 \leqslant r) \quad \ldots \quad (2)$$

$$\frac{d}{dt}P_{o,r}(t) = -(\lambda+\mu)P_{o,r}(t) + \mu P_{o,r+1}(t) + \mu C_r \sum_{m=1}^{S} P_{m,1}(t) \quad (1 \leqslant r) \quad \ldots \quad (3)$$

$$\frac{d}{dt}P_o(t) = \mu P_{o,1}(t) \quad \ldots \quad (4)$$

Let the Laplace transform of $P_{n,r}(t)$ be given by $\overline{P_{n,r}}(s)$, i.e.

$$\overline{P}_{n,r}(s) = \int_0^\infty P_{n,r}(t)\, e^{-st}\, dt$$

Applying the Laplace transform to the above equations and using the initial condition mentioned above, we get

$$-(s+\lambda+\mu)\overline{P}_{n,r}(s) + \lambda\overline{P}_{n-1,r}(s) + \mu\overline{P}_{n,r+1}(s) + \mu Cr\,\overline{P}_{n+S,1}(s) = 0 \quad (n > 0, 1 \leqslant r) \quad \ldots \quad (5)$$

$$-(s+\lambda+\mu)\overline{P}_{o,r}(s) + \mu\overline{P}_{o,r+1}(s) + \mu C_r \sum_{m=1}^{S} \overline{P}_{m,1}(s) + C_r = 0 \quad (1 \leqslant r) \quad \ldots \quad (6)$$

$$s\overline{P}_o(s) - \mu\overline{P}_{o,1}(s) = 0 \quad \ldots \quad (7)$$

Let us define the following generation functions:

$$Q_n(x,s) = \sum_{r=1}^{\infty} x^r\, \overline{P}_{n,r}(s)$$

$$\overline{F}(x,y,s) = \sum_{n=0}^{\infty} y^n\, \overline{Q}_n(x,s)$$

Multiplying the equations (5) to (7) with appropriate powers of $x$ and $y$ we obtain:

$$-[s+\lambda+\mu-\lambda y-\mu/x]\,\overline{F}(x,y,s) - \frac{\mu C(x)}{y^S}\,\overline{P}_{o,1}(s)$$

$$+\mu\left[1-\frac{C(x)}{y^S}\right]\sum_{n=o}^{\infty} y^n\,\overline{P}_{n,1}(s) + C(x)$$

$$+ \mu C(x) \sum_{m=1}^{S-1} \overline{P}_{m,1}(s)\,[1 - y^m / y^S] = o \quad \ldots \quad \ldots \quad (8)$$

Putting $x = \mu/(\mu + s + \lambda - \lambda y)$, we get

$$\sum_{n=o}^{\infty} y^n \overline{P}_{n,1}(s) = \frac{\mu \sum_{m=1}^{S-1} \overline{P}_{m,1}(s)(y^S - y^m) - \mu \overline{P}_{o,1}(s) + y^S}{\mu\,[y^S / B(y) - 1]} \quad (9)$$

where

$$B(y) = \sum_{r=1}^{\infty} C_r \left( \frac{\mu}{\mu + s + \lambda - \lambda y} \right)^r$$

It has been shown by the author[13] that the denominator of (9) has S zeros inside the unit circle and at these zeros the numerator of (9) must vanish so that $\sum_{n=o}^{\infty} y^n \overline{P}_{n,1}(s)$ will be regular for Re $s > 0$. Thus the S unknown in (9) will be determined by the S equations, namely

$$\mu \overline{P}_{o,1}(s) + \mu \sum_{m=1}^{S-1} \overline{P}_{m,1}(s) \left( y_i^m - y_i^S \right) = y_i^S \quad (10)$$

$$(i = 1, 2 \ldots\ldots\ldots s)$$

so that

$$\mu \overline{P}_{o,1}(s) = \frac{|A|}{|B|}$$

where

$$|A| = \begin{vmatrix} y_1^S & y_1 & y_1^2 & \cdot\cdot & \cdot\cdot & y_1^{S-1} \\ y_2^S & y_2 & y_2^2 & \cdot\cdot & \cdot\cdot & y_2^{S-1} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ y_S^S & y_S & y_S^2 & \cdot\cdot & \cdot\cdot & y_S^{S-1} \end{vmatrix}$$

$$= (-1)^{S-1}\, y_1 y_2 \ldots y_S \,\Pi\, (y\mu - y\nu)$$

$$\mu \neq \nu,\ \mu < \nu \quad (\mu, \nu = 1, 2 \ldots S)$$

and

$$|B| = \begin{vmatrix} 1 & y_1 - y_1^S & \cdots & \cdots & y_1^{S-1} - y_1^S \\ 1 & y_2 - y_2^S & \cdots & \cdots & y_2^{S-1} - y_2^S \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & y_S - y_S^S & \cdots & \cdots & y_S^{S-1} - y_S^S \end{vmatrix}$$

$$= \left[ \prod_{i=1}^{S} (1 - y_i) + (-1)^{S-1} y_1 y_2 \ldots y_S \right] \prod_{\mu \neq \nu} (y_\mu - y_\nu)$$

$$\mu < \nu \; (\mu, \nu = 1, 2 \ldots S)$$

Hence if $|B| \neq o$, which is a necessary and sufficient condition that the equations (10) may have a unique solution, then

$$\mu \overline{P}_{o,1}(s) = \frac{(-1)^{S-1} y_1 y_2 \ldots y_S}{\prod_{i=1}^{S} (1 - y_i) + (-1)^{S-1} y_1 y_2 \ldots y_S} \qquad \ldots (12)$$

Hence the Laplace transform of the required probability density is given by

$$\bar{r}(s) = \mu \overline{P}_{o,1}(s) = \frac{1}{1 + (-1)^{S-1} \prod_{i=1}^{S} \frac{1 - y_i}{y_i}} \qquad \ldots \ldots (13)$$

where $y_i$ are the S roots of modulus less than one of the equation

$$y^S - \sum_{r=1}^{\infty} C_r \left( \frac{\mu}{\mu + s + \lambda - \lambda y} \right)^r = o \qquad \ldots \ldots (14)$$

However, if the number of roots of modulus greater than one is less than S, it would be preferable to express $\bar{\gamma}(s)$ in terms of the roots of modulus greater than one. Let us consider, for example, the case in which the service time distribution is K-Erlang, where $K < s$. Then if $|B| \neq o$, we have from (9)

$$\sum_{n=o}^{\infty} y^n P_{n,1}(s) = \frac{d(s)}{\prod_{i=s+1}^{s+k} (y_i - y)} \qquad \ldots \ldots (15)$$

so that

$$\overline{P}_{o,1}(s) = d(s) \Big/ \prod_{i=s+1}^{s+k} \qquad \ldots \ldots (16)$$

Also from (9)

$$\sum_{n=0}^{\infty} \bar{P}_{n,1}(s) = \frac{1-\mu \bar{P}_{o,1}(s)}{\mu\left[\left(\frac{\mu+s}{\mu}\right)^k - 1\right]} \qquad (17)$$

so that

$$\frac{\bar{P}_{o,1}(s) \prod_{i=s+1}^{s+k} \hat{y}_i}{\prod_{i=s+1}^{s+k} (\hat{y}_i - 1)} = \frac{1-\mu \bar{P}_{o,1}(s)}{\mu\left[\left(\frac{\mu+s}{\mu}\right)^k - 1\right]} \qquad (18)$$

This gives $\bar{P}_{o,1}(s)$. Hence, using (1) and (7), the Laplace transform of the required probability density is given by

$$\bar{r}(s) = \frac{1}{1 + \left[\left(\frac{\mu+s}{\mu}\right)^k - 1\right] \prod_{i=S+1}^{S+k} \left(\frac{\hat{y}_i}{\hat{y}_i - 1}\right)} \qquad (19)$$

where $\hat{y}_i$ are the $k$ roots of modulus greater than unity of the equation

$$y^S \left[\frac{\mu + s + \lambda - \lambda y}{\mu}\right]^k - 1 = o \qquad (20)$$

**Particular Cases**

(*a*) Exponential Service Time Distribution—Inversion being difficult in the general case, we consider the particular case in which the service time distribution is also exponential. For the exponential distribution, $C_r = 1, r = 1$ and $C_r = o, r \neq 1$ and $|B| \neq o$ if $s$ is chosen in such a way that $|s| > \mu(\rho - 1)$ where $\rho = \frac{\lambda k}{s\mu}$, because as shown by the author[13] under this condition no two roots are equal, none is equal to one. Also since the greatest root $\hat{y}_{S+1}$ cannot be equal to $\left(\frac{\mu}{\mu+s}\right)$, we have

$$|B| = \frac{(\mu+s)\hat{y}_{S+1} - \mu}{\lambda \hat{y}_{S+1}(\hat{y}_{S+1} - 1)} \prod_{\mu \neq v} (y\mu - yv) \neq o$$

$$\mu < v \quad (\mu, v = 1, 2 \ldots\ldots S)$$

The solution is, therefore, unique and from (19) is given by

$$\bar{r}(s) = \frac{\mu(\hat{y}_{S+1} - 1)}{(\mu+s)\hat{y}_{S+1} - \mu} \qquad (21)$$

where $\hat{y}_{S+1}$ is the root of modulus greater than unity of the equation

$$\lambda \hat{y}^{S+1} - (\mu + s + \lambda) y^S + \mu = o \qquad (22)$$

Inverting (21), we get by following Luchak[8], the required probability density as

$$\gamma(t) = \sum_{n=}^{\infty} e^{-\mu t} \int_{0}^{t} \frac{e^{-\lambda u}}{u} \frac{(t-u)^n}{n!} \left[ n \frac{I^S_n (2b\lambda u)}{b^n} - (n+1) \frac{I^S{}_{n+1} (2b\lambda u)}{b^{n+1}} \right] du \quad .. \quad (23)$$

where

$$b^{S+1} = \frac{\mu}{\lambda}$$

The moments of the distribution can be more easily evaluated from (21) by differentiation at $s=o$. For example the mean E($t$) and the variance V($t$) of the distribution of the length of busy periods are given by

$$E(t) = \frac{\hat{\alpha}_{S+1}}{\mu(\hat{\alpha}_{S+1} - 1)} \quad .. \quad .. \quad .. \quad .. \quad (24)$$

$$V(t) = \frac{2\hat{\alpha}_{S+1}}{\mu(\hat{\alpha}_{S+1} - 1)^2 \, [(S+1)\lambda\hat{\alpha}_{S+1} - (\lambda+\mu) S]} + \frac{\hat{\alpha}^2_{S+1}}{\mu^2 (\hat{\alpha}_{S+1} - 1)^2} \quad .. \quad (25)$$

where $\hat{\alpha}_{S+1}$ is obtained by putting $s = o$ in $\hat{\eta}_{S+1}$ i.e. $\hat{\alpha}_{S+1}$ is the root of modulus greater than unity of the equation

$$\lambda y^{S+1} - (\lambda + \mu) y^S + \mu = o$$

(*b*) Classical Queue—For the classical queue, S = 1 and from (13), we get

$$\bar{r}(s) = y_1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (26)$$

where $y_1$ is the unique root of modulus less than one of the equation (14) with $S = 1$.

Thus the Laplace transform of the probability density of the busy period distribution for the classical single server queue is given as the unique solution of the equation $y = \bar{\beta}[s + \lambda(1 - y)]$ where $\bar{\beta}(s)$ is the Laplace transform of the service time density—an important result due to Takács[4].

Putting $Z = \dfrac{\mu}{\mu + s + \lambda - \lambda y}$, (26) can be written as

$$\bar{r}(s) = \frac{\mu + s + \lambda}{\mu} - \frac{\mu}{\lambda Z_1} \quad .. \quad .. \quad .. \quad (27)$$

where $Z_1$ is the root of modulus less than one of the equation

$$(\lambda + \mu + s) Z - \mu - \lambda Z \sum_{r=1}^{\infty} Cr \, Z^r = o \quad .. \quad .. \quad (28)$$

Now proceeding as in Luchak[8], we get

$$Z_1^{-1} = \left(\frac{\mu}{\lambda+\mu+s}\right)^{-1} - \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n}\frac{1}{n!}\sum_{r=o}^{\infty} b_{n,r}\,\frac{(r+2n-2)!}{(r+n-1)!}\times \left(\frac{\mu}{\lambda+\mu+s}\right)^{r+2n-1} \quad .. \quad .. \quad (29)$$

where

$$\sum_{r=1}^{\infty}\left(C_r\, Z^{r-1}\right)^{n} = \sum_{r=o}^{\infty} b_{n,r}\; Z^{r} \quad .. \quad .. \quad (30)$$

Substituting (29) in (27) and inverting, we finally get

$$r\,(t) = \frac{\mu}{\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n}}{n!}\sum_{r=o}^{\infty} b_{n,r}\,\frac{(\mu t)^{r+n-1}}{(r+n-1)!}\, e^{-(\lambda+\mu)t} \quad .. \quad (31)$$

It will be observed that if $Bn(t)$ represents the $n$-fold convolution of $B(t)$, then using (30), we get

$$d\,B_n\,(t) = \sum_{r=o}^{\infty} b_{n,r}\,\frac{(\mu t)^{r+n-1}}{(r+n-1)!}\, e^{-\mu t}\,\mu\, dt \quad .. \quad .. \quad (32)$$

and therefore (31) can be written as

$$r\,(t)\,dt = \sum_{n=1}^{\infty} e^{-\lambda t}\,\frac{(\lambda t)^{n-1}}{n!}\, d\,B_n\,(t) \quad .. \quad .. \quad .. \quad (33)$$

which is equation (9) of Prabhu's paper referred above.

The table below gives the mean length of the busy periods in terms of the mean arrival time for different values of S and $\rho$, assuming exponential service time distribution.

| $\rho$ | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 |
|---|---|---|---|---|---|---|---|---|---|
| S=1 | ·111 | ·250 | ·429 | ·667 | 1·000 | 1·500 | 2·333 | 4·000 | 9·000 |
| S=2 | ·241 | ·577 | 1·038 | 1·683 | 2·480 | 4·056 | 6·497 | 11·441 | 26·386 |
| S=3 | ·391 | ·983 | 1·833 | 3·056 | 4·861 | 7·675 | 12·475 | 22·334 | 52·151 |
| S=4 | ·561 | 1·468 | 2·815 | 4·785 | 7·731 | 12·359 | 20·454 | 36·673 | 86·359 |
| S=5 | ·751 | 2·035 | 3·984 | 6·873 | 11·226 | 18·106 | 30·015 | 54·063 | 128·809 |

**Acknowledgement**

The author is extremely grateful to Dr. R.S. Varma, Director, Defence Science Laboratory for guiding throughout the preparation of this paper. This paper forms part of the author's Ph. D thesis submitted to the University of Delhi.

**References**

1 Kendall, D.G., *J. Roy. Stat. Soc.* B. **13**, 151, 1951.

2 Ledermann, W. & Reuter, G.E.H., *Phil. Trans. Roy. Soc.* London, A, **246**, 321, 1954.

3 Bailey, N.T.J., *J. Roy. Stat. Soc.* B. **16**, 288, 1954.

4 Takács, L., *Acta. Math. Acad. Sci.* **6**, 101, 1955.

5 Benês, V.E., *Ann. Math. Stat.* **28**, 670, 1957.

6 McMillan, B. & Riordon, J., *Ann. Math. Stat.* **28**, 471, 1957.

7 Luchak, G., *Opns. Res.* **5**, 205, 1957.

8 Luchak, G., *J. Roy. Stat. Soc.* B. **20**, 176, 1958.

9 Conolly, B.W., *Biometrika*, **46**, 246, 1959.

10 Conolly, B.W., *J. Roy. Stat. Soc.* B. **22**, 89, 1960.

11 Prabhu, N.U., *Stat. Soc.* B. **22**, 104, 1960.

12 Bailey, N.T.J., *J. Roy. Stat. Soc.* B. **16**, 80, 1954.

13 Jaiswal, N.K., *Opns. Res.* **8**, 773, 1960.