SOME PARTICULAR SOLUTIONS OF THE EQUATION $E^4\Psi = 0$

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ABSTRACT

This paper describes some particular solutions of the equation $E^4\Psi = 0$ using the method of separating the variables.

1. Recently Bhatnagar[1] has given some particular solutions of the Biharmonic equation

$$\nabla^2 \Phi = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot1)$$

where ∇ is the Laplacian operator, using the method of separation of variables. The aim of the present note is to obtain the corresponding solutions for the equation

$$E^4\Psi = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot2)$$

where $$E^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi} \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot3)$$

which occurs in the discussion of the axis-symmetric motion of viscous liquids.

In terms of the polar co-ordinates,

$$x = r\mu, \ \varpi = r\sqrt{1-\mu^2}, \ \mu = \cos\theta \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot4)$$

we have

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} \quad \ldots \quad \ldots \quad \ldots \quad (1\cdot5)$$

The technique of solving the Laplace equation by separating the variables is well known. We shall use this technique to solve $E^4\Psi = 0$ in two sets of co-ordinates (x, ϖ) and (r, μ).

2. In (x, ϖ) co-ordinates (1·2) can be written as

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\right]^2 \psi = 0 \quad \ldots \quad (2\cdot1)$$

i.e.

$$\frac{\partial^4\Psi}{\partial x^4} + \frac{\partial^4\Psi}{\partial \varpi^4} + 2\frac{\partial^4\Psi}{\partial x^2 \partial \varpi^2} + \frac{3}{\varpi}\frac{\partial^2\Psi}{\delta \omega^2} - \frac{2}{\varpi}\frac{\partial^3\Psi}{\partial x^2 \partial \varpi}$$
$$- \frac{2}{\varpi}\frac{\partial^3\Psi}{\partial \omega^3} - \frac{3}{\varpi^3}\frac{\partial\Psi}{\partial \varpi} = 0 \quad \ldots \quad \ldots \quad (2\cdot2)$$

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On substituting $\Psi = X(x)\, W\,(\varpi)$ in (2·2) we have

$$X^{IV} + 2X'' \frac{1}{W}\left(W'' - \frac{1}{\varpi} W'\right) + X \frac{1}{W}\left[\left(\frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\right)\left(W'' - \frac{1}{\varpi} W'\right)\right] = 0 \quad \ldots \quad (2\cdot 3)$$

where dash denotes differentiation with respect to the argument.

Here we find that the separation of variables is possible in the following cases:—

(*i*) $X'' = 0$ (*ii*) $X'' = \pm \lambda^2 x$

(*iii*) $W'' - \frac{1}{\varpi} W'' = 0$ (*iv*) $W'' - \frac{1}{\varpi} W' = \alpha W$

Case (*i*) When $X'' = 0$, $X = ax + b$ .. (2·4)

and (2·3) takes the form

$$\varpi^4 W^{IV} - 2\varpi^3 W''' + 3\varpi^2 W'' - 3\varpi W'' = 0 \quad \ldots \quad (2\cdot 5)$$

Substituting $\varpi = e^z$ in (2·5) and denoting $\frac{d}{dz}$ by D we have

$$D(D - 2)^2(D - 4)\, W = 0 \quad \ldots \quad (2\cdot 6)$$

the solution of which is

$$W = C_1 + (C_2 + C_3 \log \varpi)\varpi^2 + C_4 \varpi^4 \quad \ldots \quad (2\cdot 7)$$

so that

$$\Psi = (ax + b)[C_1 + (C_2 + C_3 \log \varpi)\varpi^2 + C_4 \varpi^4] \quad \ldots \quad (2\cdot 8)$$

Hence the particular solutions of (2·1) are

$$\varpi^2,\ \varpi^4,\ x,\ x\varpi^4,\ x\varpi^2,\ \varpi^2 \log \varpi,\ x\varpi^2 \log \varpi \quad \ldots \quad (2\cdot 9)$$

Case (*ii*) When $X'' = \lambda^2 X$ .. (2·10)

then $X^{IV} = \lambda^4 X$ .. (2·11)

and the equation (2·1) takes the form

$$\left(\frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi} + \lambda^2\right)\chi = 0 \quad \ldots \quad (2\cdot 12)$$

where $\chi = \frac{d^2 W}{d\varpi^2} - \frac{1}{\varpi}\frac{dW}{d\varpi} + \lambda^2 W$ .. (2·13)

The solution of (2·12) is

$$\chi = a_1 \varpi J_1(\lambda\varpi) + a_2 \varpi Y_1(\lambda\varpi), \quad \ldots \quad (2\cdot 14)$$

where J_1 and Y_1 are Bessel functions.

In view of (2·14), (2·13) takes the form

$$\frac{d^2 W}{d\varpi^2} - \frac{1}{\varpi}\frac{dW}{d\varpi} + \lambda^2 W = a_1 \varpi J_1(\lambda\varpi) + a_2 \varpi Y_1(\lambda\varpi) \quad \ldots \quad (2\cdot 15)$$

We can easily show that this equation admits the following solution :

$$W = b_1\varpi J_1(\lambda\varpi) + b_2\varpi Y_1(\lambda\varpi) - \frac{\pi}{2\lambda^2}\varpi a_1 \Big[J_1(\lambda\varpi)\int \lambda\varpi J_1(\lambda\varpi)Y_1(\lambda\varpi)d\omega$$

$$- Y_1(\lambda\varpi)\int \lambda\varpi J_1^2(\lambda\varpi)d\omega\Big] - \frac{\pi}{2\lambda^2}\varpi a_2 \Big[J_1(\lambda\varpi)\int \lambda\varpi Y_1^2(\lambda\varpi)d\omega$$

$$- Y_1(\lambda\varpi)\int \lambda\varpi J_1(\lambda\varpi)\, Y_1(\lambda\omega)d\omega\Big] \quad \ldots \quad (2\cdot16)$$

Since the solution of (2·10) is

$$X = a_3 \cosh \lambda x + a_4 \sinh \lambda x \quad \ldots \quad (2\cdot17)$$

the particular solutions of (2·1) are

$$\varpi\begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} J_1(\lambda\varpi), \quad \begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} Y_1(\lambda\varpi),$$

$$-\frac{\pi}{2\lambda^2}\varpi\begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} J_1(\lambda\varpi)\int \lambda\varpi J_1(\lambda\varpi)Y_1(\lambda\varpi)d\omega,$$

$$-\frac{\pi}{2\lambda^2}\varpi\begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} J_1(\lambda\varpi)\int \lambda\varpi Y_1^2(\lambda\varpi)\, d\varpi,$$

$$+\frac{\pi}{2\lambda^2}\varpi\begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} Y_1(\lambda\varpi)\int \lambda\varpi J_1^2(\lambda\varpi)d\omega$$

$$\text{and } +\frac{\pi}{2\lambda^2}\varpi\begin{pmatrix}\text{Cosh}\,\lambda x\\ \text{Sinh}\,\lambda x\end{pmatrix} Y_1(\lambda\varpi)\int \lambda\varpi J_1(\lambda\varpi)\, Y_1(\lambda\varpi)d\varpi \quad \ldots \quad (2\cdot18)$$

When $X'' = -\lambda^2 X$ .. .. (2·19)

$$X = a_3 \cos \lambda x + b_3 \sin \lambda x \quad \ldots \quad (2\cdot20)$$

and (2·1) reduces to

$$\frac{d^2\chi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \lambda^2\chi = 0 \quad \ldots \quad (2\cdot21)$$

$$\text{where } \chi = \frac{d^2W}{d\varpi^2} - \frac{1}{\varpi}\frac{dW}{d\varpi} - \lambda^2 W \quad \ldots \quad (2\cdot22)$$

The solution of (2·21) is

$$\chi = a_1\varpi I_1(\lambda\varpi) + a_2\varpi K_1(\lambda\varpi) \quad \ldots \quad (2\cdot23)$$

In view of (2·23), (2·22) takes the form

$$\frac{d^2W}{d\varpi^2} - \frac{1}{\varpi}\frac{dW}{d\varpi} - \lambda^2 W = a_1\varpi I_1(\lambda\varpi) + a_2\varpi K_1(\lambda\varpi) \quad (2\cdot24)$$

which has the following complete solution

$$W = b_1\varpi I_1(\lambda\varpi) + b_2\varpi K_1(\lambda\varpi) + \frac{\varpi}{\lambda^2}a_1\Big[I_1(\lambda\varpi)\int \lambda\varpi I_1(\lambda\varpi)K_1(\lambda\omega)d\hat{\omega}$$

$$- K_1(\lambda\varpi)\int \lambda\, I_1^2(\lambda\ )d\omega\Big] + \frac{\omega}{\lambda^2}a_2\Big[I_1(\lambda\omega)\int \lambda\varpi K_1^2(\lambda\varpi)d\hat{\omega}$$

$$- K_1(\lambda\varpi)\int \lambda\varpi I_1(\lambda\varpi)K_1(\lambda\varpi)d\varpi\Big] \quad \ldots \quad (2\cdot25)$$

In this case then we have the following particular solutions :—

$$\varpi \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} I_1 (\lambda \varpi) , \varpi \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} K_1 (\lambda \varpi) ,$$

$$\frac{\varpi}{\lambda^2} \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} I_1 (\lambda \varpi) \int \lambda \varpi I_1 (\lambda \varpi) K_1 (\lambda \varpi) \, d\omega,$$

$$\frac{\varpi}{\lambda^2} \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} I_1 (\lambda \varpi) \int \lambda \varpi K_1^2 (\lambda \varpi) d\omega - \frac{\varpi}{\lambda^2} \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} K_1 (\lambda \varpi) \int \lambda \varpi I_1^2 (\lambda \varpi) \, d\omega$$

$$\text{and} - \frac{\varpi}{\lambda^2} \begin{pmatrix} \operatorname{Cos} \lambda x \\ \operatorname{Sin} \lambda x \end{pmatrix} K_1 (\lambda \varpi) \int \lambda \varpi I_1 (\lambda \varpi) K_1 (\lambda \varpi) d\varpi \qquad (2 \cdot 26)$$

Case (*iii*) When $W'' - \dfrac{1}{\varpi} W' = 0$ .. .. (2·27)

the equation (2·3) reduces to

$X^{IV} = 0$ having the complete solution

$$X = a + bx + cx^2 + dx^3 \qquad (2 \cdot 28)$$

Since the solution of (2·27) is

$$W = a_1 + a_2 \varpi^2 \qquad (2 \cdot 29)$$

$$\Psi = (a_1 + a_2 \varpi^2) \, [\, a + bx + cx^2 + dx^3 \,] \qquad (2 \cdot 30)$$

giving the following particular solutions :—

$$x, \, x\varpi^2, \, x^2, \, x^2\varpi^2, \, x^3, \, x^3 \varpi^2, \, \varpi^2 \qquad (2 \cdot 31)$$

Case (*iv*) Here $W'' - \dfrac{1}{\varpi} W' = a W$ .. .. .. (2·32)

where $a = \pm \lambda^2$ .. .. (2·33)

When $a = \lambda^2$ then $W'' - \dfrac{1}{\varpi} W' = \lambda^2 W$ .. .. (2·34)

$$\text{so that} \left(\frac{d^2}{d\varpi^2} - \frac{1}{\varpi} \frac{d}{d\varpi} \right) \left(W'' - \frac{1}{\varpi} W' \right) = \lambda^4 W \qquad (2 \cdot 35)$$

and the equation (2·3) takes the form

$$X^{IV} + 2 \lambda^2 X'' + \lambda^4 X = 0 \qquad (2 \cdot 36)$$

the complete solution of which is

$$X = (b_1 + b_2 x) \operatorname{Cos} \lambda x + (b_3 + b_4 x) \operatorname{Sin} \lambda x \qquad (2 \cdot 37)$$

Further the solution of (2·34) is

$$W = a_1 \varpi I_1 (\lambda \varpi) + a_2 \varpi K_1 (\lambda \varpi) \qquad (2 \cdot 38)$$

(2·37) and (2·38) give us the following particular solutions of (2·1) :—

$$\varpi I_1 (\lambda \varpi) \operatorname{Cos} \lambda x, \varpi K_1 (\lambda \varpi) \operatorname{Cos} \lambda x, \varpi x I_1 (\lambda \varpi) \operatorname{Cos} \lambda x, \varpi x K_1 (\lambda \varpi) \operatorname{Cos} \lambda x,$$

$$\varpi I_1 (\lambda \varpi) \operatorname{Sin} \lambda x, \varpi K_1 (\lambda \varpi) \operatorname{Sin} \lambda x, \varpi x I_1 (\lambda \varpi) \operatorname{Sin} \lambda x, \varpi x K_1 (\lambda \varpi) \operatorname{Sin} \lambda x, \qquad (2 \cdot 39)$$

Let us now have $W'' - \dfrac{1}{\varpi} W' = -\lambda^2 W$ .. .. (2·40)

then (2·3) reduces to

$X^{IV} - 2\lambda^2 X'' + \lambda^4 X = 0$ .. .. .. (2·41)

the solution of which is

$X = (b_1 + b_2 x) \operatorname{Cosh} \lambda x + (b_3 + b_4 x) \operatorname{Sinh} \lambda x$ .. .. (2·42)

Further the solution of (2·40) is

$W = a_1 \varpi J_1 (\lambda\varpi) + a_2 \varpi Y_1 (\lambda\varpi)$ .. .. .. (2·43)

From (2·42) and (2·43) we have the following particular solutions :—

$$\varpi \begin{pmatrix} \operatorname{Cosh} \lambda x \\ \operatorname{Sinh} \lambda x \end{pmatrix} J_1 (\lambda\varpi), \quad \varpi \begin{pmatrix} \operatorname{Cosh} \lambda x \\ \operatorname{Sinh} \lambda x \end{pmatrix} Y_1 (\lambda\varpi),$$

$$\varpi x \begin{pmatrix} \operatorname{Cosh} \lambda x \\ \operatorname{Sinh} \lambda x \end{pmatrix} J_1 (\lambda\varpi), \quad \varpi x \begin{pmatrix} \operatorname{Cosh} \lambda x \\ \operatorname{Sinh} \lambda x \end{pmatrix} Y_1 (\lambda\varpi) \qquad (2·44)$$

3. In polar co-ordinates defined in (1·4) the equation (1·2) reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2} \right]^2 \Psi = 0 \quad .. \quad .. \quad (3·1)$$

Taking $\Psi = R(r) M(\mu)$ .. .. (3·2)

in (3·1) we have

$$\left[\frac{r^4 R^{IV}}{R} \right] + 2 \left[(1-\mu^2) \frac{M''}{M} \right] \left[\frac{r^2 R'' - 2rR' + 3R}{R} \right]$$
$$+ \left[\frac{1-\mu^2}{M} \frac{d^2}{d\mu^2} \left\{ (1-\mu^2) M'' \right\} \right] = 0 \quad .. \quad (3·3)$$

Substituting $r = e^z$ in (3·3) and denoting $\dfrac{d}{dz}$ by D,

we have

$$\left[\frac{1}{R} D(D-1)(D-2)(D-3) R \right] + 2 \left[(1-\mu^2) \frac{M''}{M} \right] \left[\frac{1}{R} (D^2 - 3D + 3) R \right]$$
$$+ \left[\frac{1-\mu^2}{M} \frac{d^2}{d\mu^2} \left\{ (1-\mu^2) M'' \right\} \right] = 0 \quad .. \quad (3·4)$$

The variables are separable in (3·4) in the following cases :—

(α) $M'' = 0$ (β) $(1-\mu^2) M'' = \lambda M$

and (γ) $(D^2 - 3D + 3) R = a R$

We shall consider these cases in turn.

Case (α) : If $M'' = 0$ .. .. (3·5)

$M = a_1 + a_2 \mu$ .. .. (3·6)

and $D(D-1)(D-2)\ (D-3)R=0$

so that

$$R = b_1 + b_2 r + b_3 r^2 + b_4 r^3 \qquad (3\cdot7)$$

Hence we have the following particular solutions of (3·1) :—

$$r,\ r^2,\ r^3,\ \mu,\ r\mu,\ r^2\mu,\ r^3\mu \qquad (3\cdot8)$$

Case (β)

$$\text{Let } (1-\mu^2)M'' = \lambda M \qquad (3\cdot9)$$

$$\text{then } (1-\mu^2)\frac{d^2}{d\mu^2}\left[(1-\mu^2)M''\right] = \lambda^2 M \qquad (3\cdot10)$$

$$\text{and } [D(D-1)(D-2)(D-3) + 2\lambda(D^2-3D+3)+\lambda^2]R = O \qquad (3\cdot11)$$

To solve (3·9) in terms of the known functions we substitute

$$M = (1-\mu^2)^{\frac{1}{2}}\ N \qquad (3\cdot12)$$

in it . We then have

$$(1-\mu^2)\frac{d^2N}{d\mu^2} - 2\mu\frac{dN}{d\mu} - \left(\lambda + \frac{1}{1-\mu^2}\right)N = O \qquad (3\cdot13)$$

We now take

$$\lambda = -n(n+1) \qquad (3\cdot14)$$

so that (3·13) and (3·11) reduce to

$$(1-\mu^2)\frac{d^2N}{d\mu^2} - 2\mu\frac{dN}{d\mu} + \left[n(n+1) - \frac{1}{1-\mu^2}\right]N = O \qquad (3\cdot15)$$

and

$$[(D+n)(D-n-3)(D+n-2)(D-n-1)]R = O \qquad (3\cdot16)$$

The general solutions of (3·15) and (3·16) are

$$N = a_1 P_n{}^1(\mu) + a_2 Q_n{}^1(\mu)$$

and

$$R = b_1 e^{-nZ} + b_2 e^{(n+3)Z} + b_3 e^{-(n-2)Z} + b_4 e^{(n+1)Z}$$

so that

$$M = a_1(1-\mu^2)^{\frac{1}{2}} P_n{}^1(\mu) + a_2(1-\mu^2)^{\frac{1}{2}} Q_n{}^1(\mu) \qquad (3\cdot17)$$

and

$$R = \frac{b_1}{r^n} + b_2 r^{n+3} + b_3\frac{1}{r^{n-2}} + b_4 r^{n+1} \qquad (3\cdot18)$$

From (3·17) and (3·18) we have the following sets of particular solutions :

$$(1-\mu^2)^{1/2} P_n^{\ 1}(\mu)\frac{1}{r^{n-2}},\ (1-\mu^2)^{1/2} P_n^{\ 1}(\mu)\frac{1}{r^n},\ (1-\mu^2)^{1/2} P_n^{\ 1}(\mu)\, r^{n+1},$$

$$\text{and } (1-\mu^2)^{1/2} P_n^{\ 1}(\mu)\, r^{n+3} \quad \text{when } |\mu| < 1\,;$$

$$(1-\mu^2)^{\frac{1}{2}} Q_n^{\ 1}(\mu)\frac{1}{r^{n-2}},\ (1-\mu^2)^{\frac{1}{2}} Q_n^{\ 1}(\mu)\frac{1}{r^n},$$

$$(1-\mu^2)^{\frac{1}{2}} Q_n^{\ 1}(\mu)\, r^{n+1} \text{ and } (1-\mu^2)^{\frac{1}{2}} Q_n^{\ 1}(\mu)\, r^{n+3}$$

$$\text{when } |\mu| > 1 \qquad \ldots \qquad (3\cdot19)$$

Case (γ) :

Let $(D^2-3D+3)\, R = aR$ .. .. (3·20)

Then

$$D(D-1)(D-2)(D-3)\, R=(a-3)(a-1)\, R \quad \ldots \quad (3\cdot21)$$

and

$$(1-\mu^2)\frac{d^2}{d\mu^2}\left\{(1-\mu^2)M''\right\}+2a(1-\mu^2)M''+(a-3)(a-1)M=0 \quad (3\cdot22)$$

The discussion of the general case is cumbersome and therefore we shall only consider the special case when

$a = 1$.

When $a=1$, (3·20) and (3·22) reduce to

$$(D-1)(D-2)\, R=0 \quad \ldots \quad (3\cdot23)$$

and

$$(1-\mu^2)\frac{d^2}{d\mu^2}\left[(1-\mu^2)M''\right]+2(1-\mu^2)M''=0 \quad (3\cdot24)$$

The general solution of (3·23) is

$$R=a_1 r+a_2 r^2 \quad \ldots \quad (3\cdot25)$$

To solve (3·24) we put

$$(1-\mu^2)M''=U \quad \ldots \quad (3\cdot26)$$

so that it reduces to

$$(1-\mu^2)\frac{d^2}{d\mu^2}U+2U=0 \quad \ldots \quad (3\cdot27)$$

which is same as (3·9) on replacing M by U and taking $\lambda=-n(n+1)=-2$ i.e. $n=1$ and hence the solution of (3·27) is

$$U=b_1(1-\mu^2)^{\frac{1}{2}} P_1^{\ 1}(\mu)+b_2(1-\mu^2)^{\frac{1}{2}} Q_1^{\ 1}(\mu) \quad \ldots \quad (3\cdot28)$$

Restricting to the case when $/\mu/ < 1$, we take

$$U = b_1 (1 - \mu^2)^{\frac{1}{2}} P_1^1 (\mu) \quad \ldots \quad (3 \cdot 29)$$

so that (3·26) becomes

$$(1 - \mu^2) M'' = b_1 (1 - \mu^2)^{\frac{1}{2}} P_1^1 (\mu)$$
$$= -b_1 (1 - \mu^2),$$
$$\text{or} \quad M'' = -b_1$$
$$\text{so that} \quad M = -b_1 \frac{\mu^2}{2} + b_2 \mu + b_3 \quad \ldots \quad (3 \cdot 30)$$

From (3·25) and (3·30) we have the following new solutions :

$$r\mu^2, \quad r^2\mu^2$$

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Reference

(1) Bhatnagar, P.L. ; Lectures on Partial Differential Equations, Indian Institute of Science, Bangalore, 1960—1961.