SOME DISTRIBUTIONS ARISING FROM RANDOM DIVISION OF AN INTERVAL

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ABSTRACT

The paper deals with the distribution of certain functions of distances between any two points of a line of length unity divided into $(n + 1)$ parts by n points taken at random by a unified approach.

Introduction

If n points are distributed at random on a line of length unity, the line will be divided into $(n + 1)$ parts. Taking the length of the various sections to be x_1, x_2, $x_3, \ldots \ldots \ldots x_{n+1}$, the distributions of x's and functions of x's have been discussed by many authors. Whitworth[1] in his book 'Choice and Chance' has given the expectation for $x_1^p \; x_2^q \; x_3^r \; \ldots\ldots$ where each of the indices $p, q, r \ldots.$ is zero or a positive integer. Moran[2] has given the distribution of $S = \sum_{i=1}^{n+1} x^2_i$. He has obtained the first four moments of this distribution. Darling[3] has worked out the first two moments of the distribution of the random variable $W_n = \Sigma^{\alpha} x_i$ by applying the method of characteristic function. Barton and David[4] have discussed the joint distribution of $g_1, g_2, g_3, \ldots g_{n+1}$ where the g's are the ordered values of the x's by using the methods developed by Whitworth[1]. The distributions of $g_1, g_2, g_3, \ldots\ldots$ etc. obtained by them reduce to

$$\left.\begin{aligned} p(g_1) &= (n+1)n(1 - \overline{n+1}\, g_1)^{n-1} \\ p(g_1, g_2) &\propto \{1 - g_1 - ng_2\}^{n-2} \\ p(g_1, g_2, g_3) &\propto \{1 - g_1 - g_2 - (n-1)g_3\}^{n-3} \end{aligned}\right\} \quad (1)$$

These authors have also considered a number of other distributions formed on the basis of rth and sth smallest interval. This paper discusses the distribution of x's and its functions by a unified procedure developed by Iyer[5]. The first two moments of the distribution for the intervals between the ith and $(i + k)$ th point on the line or some functions of these intervals have also been obtained.

Distribution of intervals between successive points

(a) *Distribution and moments*

The probability of any of the interval lying between z and $z + dz$ is

$$n(1 - z)^{n-1}\, dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

This can be established as follows: An interval of length z can be either at the extreme ends or in the centre of the line. Assuming the first point to lie between z and $z + dz$ from the left end and the remaining $(n—1)$ points to lie in the region of

$(1-z)$, the probability for the fulfilment of these conditions is obviously $n(1-z)^{n-1}dz$. In a similar manner it can be shown that this result holds good to right extreme end also. If the interval is at the centre, we take two successive observation y_{i+1} and y_i. Consider the probability that $(y_{i+1}-y_i)$ is between z and $z+dz$. The probability for y_{i+1} and y_i to lie between y_{i+1} and $y_{i+1}+dy_{i+1}$ and y_i and y_i+dy_i (i taking values 1 to n) is

$$n(1-\overline{y_{i+1}-y_i})^{n-2}\,dy_i\,dy_{i+1}{}^*$$

Taking $y_{i+1}-y_i=z$, the above reduces to $n(1-z)^{n-2}\,dy_i\,dz$ (3)
Now limits of y_i are 0 to $(1-z)$ and therefore integrating (3) between these limits for y_i, the probability for z to lie between z and $z+dz$ is

$$n(1-z)^{n-1}\,dz \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The rth moment of the distribution

$$\mu'_r = n\int_0^1 z^r\,(1-z)^{n-1}\,dz = \frac{1}{\binom{n+r}{r}} \quad \ldots \quad (5)$$

(b) Expected number of intervals between y_1 and y_2 and its variance

There are $(n+1)$ intervals on the line. Now the expected number of intervals lying between y_1 and y_2 is

$$E\left\{N(y_1, y_2)\right\} = (n+1)n\int_{y_1}^{y_2}(1-z)^{n-1}\,dz$$

$$= (n+1)\left\{(1-y_1)^n-(1-y_2)^n\right\} = (n+1)\,\alpha \text{ say} \quad \ldots \quad \ldots \quad (6)$$

Using the methods developed by Iyer[5] the second factorial moment is the expectation for two of the intervals between the lengths y_1 and y_2 among the $(n+1)$ intervals on the line. The probability for any two of the intervals to be lying between z_1 and z_1+dz_1 and z_2 and z_2+dz_2 reduces to

$$n(n-1)(1-z_1-z_2)^{n-2}\,dz_1dz_2 \quad \ldots \quad \ldots \quad (7)$$

Similarly the probability for any three of the intervals to lie between z_1 and z_1+dz_1, z_2 and z_2+dz_2 and z_3 and z_3+dz_3 and also the probability for any four of the intervals to lie between z_1 and z_1+dz_1, z_2 and z_2+dz_2, z_3 and z_3+dz_3 and z_4 and z_4+dz_4 are given by

$$n(n-1)(n-2)(1-z_1-z_2-z_3)^{n-3}\,dz_1dz_2dz_3 \quad \ldots \quad (8)$$

and

$$n(n-1)(n-2)(n-3)(1-z_1-z_2-z_3-z_4)^{n-4}\,dz_1dz_2dz_3dz_4 \quad (9)$$

It may be noted that this result holds good whether the intervals are connected or disconnected.

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*This can be seen by taking the distribution to be

$$c(1-\overline{y_{i+1}-y_i})^{n-2}\,dy_i\,dy_{i+1} \text{ and equating}$$

$$c\int_0^1\int_0^{y_i}(1-\overline{y_{i+1}-y_i})^{n-2}\,dy_i\,dy_{i+1}=1$$

$\mu'_{[2]}$ = the No. of ways of having two intervals

$\times$ the probability for obtaining them

$$= (n+1)n \cdot n(n-1) \int_{y_1}^{y_2}\int_{y_1}^{y_2} (1 - z_1 - z_2)^{n-2}\, dz_1 dz_2$$

$$= n^2(n^2-1)\{(1-2y_1)^n - 2(1-y_1-y_2)^n + (1-2y_2)^n\}/n(n-1)$$

$$= n(n+1)\beta_{21} \quad \ldots \quad (10)$$

Assuming y_1 and y_2 to be $< \frac{1}{2}$, the variance

$$V\{N(y_1, y_2)\} = (n+1)\,\alpha\,(1-\alpha) + n(n+1)(\beta_{21} - \alpha^2)$$

When $y_1 = a/(n+1)$ and $y_2 = b/(n+1)$

$$E\left\{N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\} \sim (n+1)(e^{-a} - e^{-b})$$

$$V\left\{N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\} \sim (n+1)[(e^{-a} - e^{-b}) - (e^{-a} - e^{-b})^2 - (ae^{-a} - be^{-b})^2] \quad \ldots \quad (11)$$

It may be noted that the variance given here is not the same as that given by Darling[3].*

(c) *Distribution of* W_{n+1}

Darling[3] has given the characteristic function of

$$W_{n+1} = \sum_{j=1}^{n+1} h(x_j)$$

By differentiating the characteristic function he has obtained the first two moments of W_{n+1}. The first four moments of W_{n+1} are obtained here by the simplified method. It can easily be seen that

$$E(W_{n+1}) = (n+1)\,E\{h(x_j)\}$$
$$= (n+1)\,n \int_0^1 h(z)\,(1-z)^{n-1}\,dz \quad \ldots \quad (12)$$

$$E(W^2_{n+1}) = (n+1)\,E\{h(x_j)\}^2 + (n+1)\,n\,E\{h(x_j)\,h(x_e)\}$$
$$= (n+1)\,n \int_0^1 \{h(z)\}^2 (1-z)^{n-1}\,dz$$
$$+ n^2(n^2-1) \iint_{z_1+z_2 \leqslant 1} h(z_1)\,h(z_2)\,(1-z_1-z_2)^{n-2}\,dz_1\,dz_2 \quad \ldots \quad (13)$$

$$E(W^3_{n+1}) = (n+1)\,E\{h(x_j)\}^3 + 3(n+1)\,n\,E\{h(x_j)\}^2\{h(x_l)\}$$
$$+ (n+1)\,n\,(n-1)\,E\{h(x_j)\,h(x_i)\,h(x_k)\} \quad \ldots \quad (14)$$

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*This difference is due to the higher degree of approximation carried out in the present investigation.

$$= (n+1)\, n \int_0^1 \{ h(z) \}^3 (1-z)^{n-1} dz$$

$$+ 3n(n+1)n(n-1) \iint\limits_{z_1+z_2 \leqslant 1} \{h(z_1)\}^2 h(z_2) (1-z_1-z_2)^{n-2} dz_1\, dz_2$$

$$+ (n+1)n(n-1)n(n-1)(n-2) \iiint\limits_{z_1+z_2+z_3 \leqslant 1} h(z_1)\, h(z_2)\, h(z_3)$$

$$(1-z_1-z_2-z_3)^{n-3} dz_1\, dz_2\, dz_3 \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$E(W^4_{n+1}) = (n+1)\, n \int_0^1 \{h(z)\}^4 (1-z)^{n-1} dz$$

$$+ 4(n+1)\, n.\, n(n-1) \iint\limits_{z_1+z_2 \leqslant 1} \{h(z_1)\}^3 h(z_2)(1-z_1-z_2)^{n-2} dz_1\, dz_2$$

$$+ 3(n+1)\, n.\, n(n-1) \iint\limits_{z_1+z_2 \leqslant 1} \{h(z_1)\}^2 \{h(z_2)\}^2 (1-z_1-z_2)^{n-2} dz_1\, dz_2$$

$$+ \frac{12 \times 3}{6} (n+1)\, n(n-1)\, n(n-1)(n-2) \iiint\limits_{z_1+z_2+z_3 \leqslant 1}$$

$$\{h(z_1)\}^2 h(z_2) h(z_3)(1-z_1-z_2-z_3)^{n-3} dz_1 dz_2 dz_3$$

$$+ (n+1)n(n-1)(n-2)n(n-1)(n-2)(n-3) \iiiint\limits_{z_1+z_2+z_3+z_4 \leqslant 1}$$

$$h(z_1)\, h(z_2)\, h(z_3)\, h(z_4)\, (1-z_1-z_2-z_3-z_4)^{n-4} dz_1\, dz_2\, dz_3\, dz_4 \quad (16)$$

The value of the various moments for $W_{n+1} = \sum_{j=1}^{n+1} x_j^{\alpha}$ reduces to

$$\mu'_1(W_{n+1}) = \frac{(n+1)!\, \alpha!}{(n+\alpha)!} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\mu'_2(W_{n+1}) = \frac{(n+1)!}{(n+2\alpha)!} \left\{ (2\alpha)! + n(\alpha!)^2 \right\} \quad \ldots \quad (18)$$

$$\mu'_3(W_{n+1}) = \frac{(n+1)!}{(n+3\alpha)!} \left\{ (3\alpha)! + 3n(2\alpha)!\, \alpha! + n(n-1)(\alpha!)^3 \right\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

$$\mu'_4(W_{n+1}) = \frac{(n+1)!}{(n+4\alpha)!} \left[(4\alpha)! + 4n(3\alpha)!\, \alpha! + 3n\left\{ (2\alpha)! \right\}^2 \right.$$

$$\left. + 6n(n-1)(2\alpha)!\, (\alpha!)^2 + n(n-1)(n-2)(\alpha!)^4 \right] \quad \ldots \quad (20)$$

Distribution of intervals between ith and (i+ k) th points

(a) Distribution and moments

When i is zero, the probability that the kth point is between z and $z + dz$ is given by

$$\frac{n\,!}{(k-1)\,!\,(n-k)\,!}\, z^{k-1}\,(1-z)^{n-k}\,dz \qquad \ldots \quad (21)$$

when i takes any values it can be shown that the distribution reduces to same expression

The rth moment of this distribution is

$$\mu'_r = \frac{n\,!}{(k-1)\,!\,(n-k)\,!}\int_0^1 z^{k+r-1}\,(1-z)^{n-k}\,dz$$

$$= \frac{n\,!\,(k+r-1)\,!}{(k-1)\,!\,(n+r)\,!} \qquad \ldots \quad (22)$$

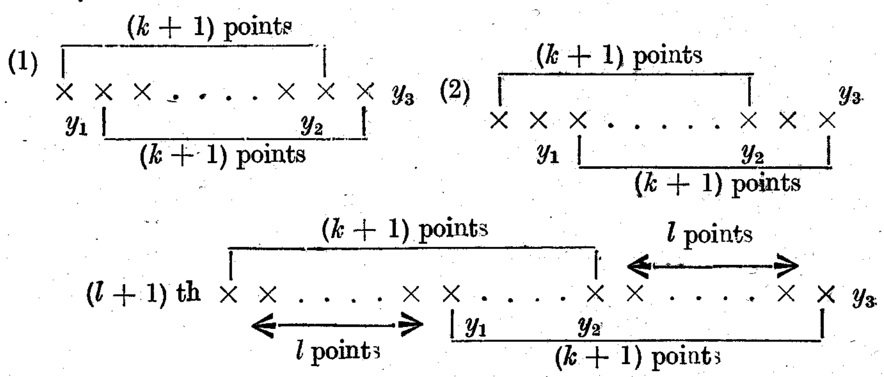
(b) Expected number of intervals between Y_1 and Y_2 and its variance

The expected number of such intervals can be seen to be equal to

$$\frac{n\,!\,(n+2-k)}{(k-1)\,!\,(n-k)\,!}\int_{Y_1}^{Y_2} z^{k-1}\,(1-z)^{n-k}\,dz \qquad \ldots \quad (23)$$

The second factorial moment is the expectation of two such intervals. Two such intervals can be (*i*) overlapping (*ii*) adjoining or (*iii*) independent ones. Taking (*i*) there are (k—1) overlapping cases.

They are



where y_1, y_2, y_3 are the distances from the beginning of the line. The probabilities of the various configurations shown above are:—

$$(1)\quad \frac{n\,!}{(k-2)\,!\,(n-k-1)\,!}\int_0^{z_1}\int_{y_1}^{z_1}\int_{y_2}^{y_1+z_2}(y_2-y_1)^{k-2}\,(1-y_3)^{n-k-1}\,dy_1\,dy_2\,dy_3 = \alpha_{o\,k} \text{ (say)} \qquad \ldots \quad (24)$$

$$(2)\quad \frac{n!}{(k-3)!\,(n-k-2)!}\int_{0}^{z_1}\int_{y_1}^{z_1}\int_{y_2}^{y_1+z_2}(y_2-y_1)^{k-3}\,y_1\,(y_3-y_2)$$

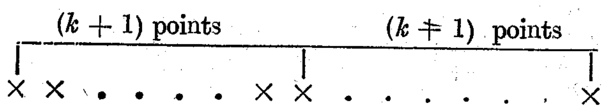
$$(1-y_3)^{n-k-2}\,dy_1\,dy_2\,dy_3=\alpha_{1k} \quad \ldots \quad (25)$$

$$\ldots \quad \ldots \quad \ldots$$

$$(l+1)\quad \frac{n!}{(k-l-2)!\,(l!)^2\,(n-k-l-1)!}\int_{0}^{z_1}\int_{y_1}^{z_1}\int_{y_2}^{y_1+z_2}(y_2-y_1)^{k-l-2}$$

$$y_1^{l}\,(y_3-y_2)^{l}\,(1-y_3)^{n-k-l-1}\,dy_1\,dy_2\,dy_3=\alpha_{lk} \quad \ldots \quad (26)$$

The limits have been obtained by assuming $y_2 \leqslant z_1$ and $y_3 - y_2 \leqslant z_2$. The probability for (*ii*) viz, the adjoining interval shown below—



reduces to

$$\frac{n!}{\{(k-1)!\}^2\,(n-2k)!}\int_{0}^{z_2}\int_{0}^{z_1} z_1^{k-1}\,z_2^{k-1}\,(1-z_1-z_2)^{n-2k}\,dz_1\,dz_2$$

$$=\beta_{2k}\,(\text{say}) \quad \ldots \quad (27)$$

When the intervals are independent, the probability that the two intervals lie between z_1 and z_1+dz_1, and z_2 and z_2+dz_2 is the same as given for the adjoining ones.

The second factorial moment of this distribution is the sum of the expectations for the above configurations assuming both z_1 and z_2 between Y_1 and Y_2. Thus

$$\frac{\mu'_{[2]}}{2!}=(n-k+1)\,\alpha_{0k}+(n-k)\,\alpha_{1k}+(n-k-1)\,\alpha_{2k}+\ldots$$

$$\ldots+(n-k-l+1)\,\alpha_{lk}+\ldots+(n-2k+3)\,\alpha_{k-2,k}$$

$$+\binom{n-2k+3}{2}\beta_{2k} \quad \ldots \quad (28)$$

The evaluation of α_{lk} for any value of k is very cumbersome and therefore we shall be contended for the present by giving the values of μ_2 for $k=2$ and 3. The values of α_{02}, and β_{22}, α_{03}, a_{13} and β_{23} are given below

$$\alpha_{02}=n\,(Y_1-Y_2)\left\{(1-Y_2)^{n-1}-(1-Y_1)^{n-1}\right\}-(1-2Y_2)^{n}$$

$$+2\,(1-Y_1-Y_2)^{n}-(1-2Y_1)^{n} \quad \ldots \quad (29)$$

$$\beta_{22} = (1-2Y_2)^n - 2(1-Y_2-Y_1)^n + (1-2Y_1)^n$$
$$+ 2n\left\{Y_2(1-2Y_2)^{n-1} - (Y_1+Y_2)(1-Y_2-Y_1)^{n-1}\right.$$
$$\left. + Y_1(1-2Y_1)^{n-1}\right\} + n(n-1)\left\{Y_2^2(1-2Y_2)^{n-2}\right.$$
$$\left. - 2Y_1Y_2(1-Y_2-Y_1)^{n-2} + Y_1^2(1-2Y_1)^{n-2}\right\} \quad .. \quad (30)$$

$$\alpha_{03} = (1-2Y_2)^n - 2(1-Y_2-Y_1)^n + (1-2Y_1)^n$$
$$+ n(Y_2-Y_1)\left\{(1-Y_2)^{n-1} - (1-Y_1)^{n-1}\right\}$$
$$- \frac{n(n-1)}{2}(Y_2^2-Y_1^2)\left\{(1-Y_2)^{n-2} - (1-Y_1)^{n-2}\right\} \quad (31)$$

$$\alpha_{13} = 5\left\{2(1-Y_2-Y_1)^n - (1-2Y_2)^n - (1-2Y_1)^n\right\}$$
$$+ n\left[4(1-Y_2-Y_1)^{n-1}(Y_2+Y_1) - 4(1-2Y_2)^{n-1}Y_2\right.$$
$$\left. - 4(1-2Y_1)^{n-1}Y_1 - 3(Y_2-Y_1)\left\{(1-Y_2)^{n-1} - (1-Y_1)^{n-1}\right\}\right]$$
$$+ n(n-1)\left[2Y_1Y_2(1-Y_2-Y_1)^{n-2} - (1-2Y_2)^{n-2}Y_2^2\right.$$
$$- (1-2Y_1)^{n-2}Y_1^2 + \left\{(1-Y_1)^{n-2} + (1-Y_2)^{n-2}\right\}$$
$$\left.\left\{Y_1Y_2 - \frac{Y_1^2}{2} - \frac{Y_2^2}{2}\right\}\right] \quad .. \quad .. \quad (32)$$

$$\beta_{23} = (1-2Y_2)^n - 2(1-Y_2-Y_1)^n + (1-2Y_1)^n$$
$$+ 2n\left\{Y_2(1-2Y_2)^{n-1} - (Y_1+Y_2)(1-Y_2-Y_1)^{n-1} + Y_1(1-2Y_1)^{n-1}\right\}$$
$$+ n(n-1)\left\{2Y_2^2(1-2Y_2)^{n-2} - (Y_1+Y_2)^2(1-Y_2-Y_1)^{n-2} + 2Y_1^2(1-2Y_1)^{n-2}\right\}$$
$$+ n(n-1)(n-2)\left\{Y_2^3(1-2Y_2)^{n-3} - Y_1Y_2(Y_1+Y_2)(1-Y_2-Y_1)^{n-3}\right.$$
$$\left. + Y_1^3(1-2Y_1)^{n-3}\right\}$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4}$$
$$\left\{Y_2^4(1-2Y_2)^{n-4} - 2Y_2^2Y_1^2(1-Y_2-Y_1)^{(n-4)} + Y_1^4(1-2Y_1)^{(n-4)}\right\} \quad .. \quad (33)$$

These values have been obtained by substituting the limits Y_1 and Y_2 for z_1 and z_2 in the general expression for the probabilities of the different configurations

Thus for $k = 2$

$$E\left\{N(Y_1,Y_2)\right\} = n \frac{n!}{(n-2)!} \int_{Y_1}^{Y_2} z(1-z)^{n-2} dz$$

$$= -n \left[n \left\{ (1-Y_2)^{n-1} Y_2 - (1-Y_1)^{n-1} Y_1 \right\} + \left\{ (1-Y_2)^n - (1-Y_1)^n \right\} \right] \quad \ldots \quad \ldots \quad (34)$$

$$= n\alpha_2$$

$$V\left\{N(Y_1,Y_2)\right\} = n\alpha_2(1-\alpha_2) + 2(n-1)(\alpha_{02} - \alpha_2^2) + (n-1)(n-2)(\beta_{22} - \alpha_2^2) \quad \ldots \quad \ldots \quad (35)$$

For $k = 3$

$$E\left\{N(Y_1,Y_2)\right\} = (n-1) \frac{n!}{(n-3)!\,2!} \int_{Y_1}^{Y_2} z^2(1-z)^{n-3} dz$$

$$= (n-1) \left[\binom{n}{2} \left\{ Y_1^2 (1-Y_1)^{n-2} - Y_2^2 (1-Y_2)^{n-2} \right\} + n \left\{ Y_1(1-Y_1)^{n-1} - Y_2(1-Y_2)^{n-1} \right\} + \left\{ (1-Y_1)^n - (1-Y_2)^n \right\} \right] \quad (36)$$

$$= (n-1)\alpha_3$$

and

$$V\left\{N(Y_1,Y_2)\right\} = (n-1)\alpha_3(1-\alpha_3) + 2(n-2)(\alpha_{03} - \alpha_3^2) + 2(n-3)(\alpha_{13} - \alpha_3^2) + (n-3)(n-4)(\beta_{23} - \alpha_3^2) \quad \ldots \quad (37)$$

Assuming $Y_1 = a/(n+1)$ and $Y_2 = b/(n+1)$ the asymptotic values of $E\left\{N(Y_1,Y_2)\right\}$ and $V\left\{N(Y_1,Y_2)\right\}$ reduce to the following.

When $k = 2$

$$E\left\{N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\} \sim n\left\{(1+a)e^{-a} - (1+b)e^{-b}\right\} \quad (38)$$

$$V\left\{N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\} \sim n\left[e^{-a}(a+1) - e^{-b}(b+1) + 2(a-b)(e^{-a} - e^{-b}) - 2\left(e^{-a} - e^{-b}\right)^2 - 3\left\{(a+1)e^{-a} - (b+1)e^{-b}\right\}^2 - \left(a^2 e^{-a} - b^2 e^{-b}\right)^2\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

For $k = 3$

$$E\left\{ N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\}$$

$$\sim (n-1)\left\{e^{-a}\left(1+a+\frac{a^2}{2}\right) - e^{-b}\left(1+b+\frac{b^2}{2}\right)\right\} \qquad (40)$$

$$V\left\{ N\left(\frac{a}{n+1}, \frac{b}{n+1}\right)\right\}$$

$$\sim (n-1)\Big[e^{-a}(1+a+a^2/_2) - e^{-b}(1+b+b^2/_2)$$

$$-2(a-b)\left\{(2+a)e^{-a} - (2+b)e^{-b}\right\}$$

$$-4\left(e^{-a} - e^{-b}\right)^2 - \frac{1}{2}\left(ae^{-a} - be^{-b}\right)^2 + \frac{5}{4}\left(a^2 e^{-a} - b^2 e^{-b}\right)^2$$

$$-\frac{1}{4}\left(a^3 e^{-a} - b^3 e^{-b}\right)^2 - 9\left\{(1+a)e^{-a} - (1+b)e^{-b}\right\}^2$$

$$-\frac{5}{2}\left\{a(1+a)e^{-a} - b(1+b)e^{-b}\right\}^2$$

$$+5(a-b)^2 e^{-(a+b)}\Big] \quad .. \quad .. \quad .. \quad .. \qquad (41)$$

(c) *Distribution of functions of intervals*

The moments for the distribution of

$$W_{k,\,n+2-k} = \sum_{1}^{n+2-k} h(x_{kj})$$

where x_{kj}, represents the distance between the jth and $(j+k)$ th point on the line, can be derived as follows.

As in the previous case

$$E(W_{k,\,n+2-k}) = (n+2-k)\, E\{h(x_{kj})\}$$

$$= \frac{(n+2-k)\, n!}{(k-1)!\,(n-k)!}\int_0^1 h(z)\, z^{k-1}(1-z)^{n-k}\, dz \quad .. \qquad (42)$$

$$E(W^2_{k,\,n+2-k}) = (n+2-k)\left[E\left\{h(x_{kj})\right\}\right]^2$$

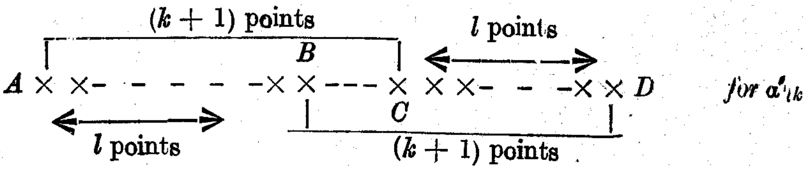
$$+\,2(n+1-k)\,\alpha'_{ok} + 2(n-k)\alpha'_{1k}$$

$$+ \,\ldots\ldots + 2(n-k-l+1)\,\alpha'_{lk} + \ldots 2(n-2k+3)\alpha'_{k-2,k}$$

$$+(n-2k+3)(n-2k+2)\beta'_{2k} \quad .. \quad .. \qquad (43)$$

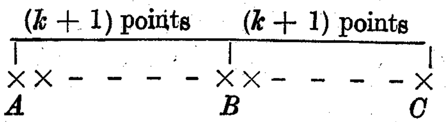
where α' and β' are obtained from the configurations given below

(1) *Overlapping*



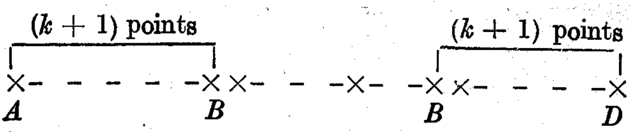
$AB = z_1$, $BC = z_2$ *and* $CD = z_3$

(2) *Adjoining*



$AB = z_1'$, $BC = z_2'$

(3) *Disconnected*



$AB = z_1'$, $CD = z_2'$.

for β'_{2k}

$$\alpha'_{lk} = \frac{n\,!}{(l!)^2\,(k-l-2)\,!\,(n-k-l-1)\,!} \iiint\limits_{z_1+z_2+z_3 \leqslant 1} z_1^{\,l}\, z_2^{\,k-l-2}$$

$$z_3^{\,l}\,(1-z_1-z_2-z_3)^{n-k-l-1}\; h(z_1+z_2)\,h(z_2+z_3)\,dz_1\,dz_2\,dz_3$$

$$\beta'_{2k} = \frac{n!}{[\,(k-1)\,!\,]^2\,(n-2k)\,!} \iint\limits_{z_1'+z_2' \leqslant 1} z_1'^{\,k-1}\; z_2'^{\,k-1}\;(1-z_1'-z_2')$$

$$h\,(z_1')\,h\,(z_2')\,dz_1'\,dz_2'$$

Taking $h\,(x_{kj}) = \Sigma\, x_{kj}^{\alpha}$, $E\,(W_{k,\,n+2-k})$ *and* $E\,(W^2_{k,\,n+2-k})$ reduce to

$$E\,(W_{k,\,n+2-k}) = \frac{n\,!\,(\alpha+k-1)\,!\,(n+2-k)}{(\alpha+n)\,!\,(k-1)\,!} \qquad \text{.. (44)}$$

$$E\,(W^2_{k,\,n+2-k}) = 2 \sum_{l=0}^{k-2} \frac{n\,!\,(\alpha\,!)^2\,(n+1-k-l)}{(n+2\alpha)\,!\,(l\,!)^2}$$

$$\sum_{r=0}^{\alpha} \sum_{s=0}^{\alpha} \frac{(\alpha-r+l)\,!\,(\alpha-s+l)\,!\,(r+s+k-l-2)\,!}{(\alpha-r)\,!\,(\alpha-s)\,!\,r\,!\,s\,!\,(k-l-2)\,!}$$

$$+ (n - 2k + 2)(n - 2k + 3) \frac{n! \left\{ (a + k - 1)! \right\}^2}{(n + 2a)! \left\{ (k - 1)! \right\}^2}$$

$$+ \frac{n! (2a + k - 1)! (n + 2 - k)}{(2a + n)! (k - 1)!} \quad \ldots \quad \ldots \quad (45)$$

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