A THEORETICAL STUDY OF THE VALIDITY OF THE ISOTHERMAL MODEL IN INTERNAL BALLISTICS

by

J. N. Kapur and B.S. Jain

University of Delhi, Delhi

ABSTRACT

The effect of the neglect of the kinetic energy term in isothermal model in internal ballistics has been discussed. The adjustments to be made in the parameters have been worked out. It has been shown that either M is to be increased by about 18·7% for tubular charges and by about 17·7% for cord charges or θ is to be decreased by about 16·0%.

Introduction

The isothermal model has been used as a simple ballistic model to represent physical phenomena inside guns. Some empirical rules regarding the adjustment of charge and design parameters, to take into account the effect of the neglect of the kinetic energy term, have been obtained in England but, for obvious reasons, these have not been published. Corner[1] has however, published some results of a comparison of theory with experiment. But no systematic theoretical study of the validity of this model or of the adjustments in parameters has been made. The object of the present paper is to fulfill partly this need.

Solution of the Basic Equations for the Isothermal Model

The four basic equations of internal ballistics for an isothermal model are :—

$$z = \zeta\xi \qquad \ldots \quad (1)$$

$$M\zeta = \eta \frac{d\eta}{d\xi} \qquad \ldots \quad (2)$$

$$\zeta = -\eta \frac{df}{d\xi} \qquad \ldots \quad (3)$$

$$z = (1-f)(1+\theta f), \qquad \ldots \quad (4)$$

where

$$\left.\begin{aligned} \xi &= 1 + \frac{x}{l} \\ \eta &= \frac{AD}{F\rho C} v \\ \zeta &= \frac{Al}{FC} p \\ M &= \frac{A^2D^2}{F\rho^2 C W_1} \end{aligned}\right\} \qquad \ldots \quad (5)$$

The initial conditions at the shot-start are $\xi = 1$, $\eta = 0$. We take the shot start pressure to be zero i.e. we assume $f_o = 1$.

From (2) and (3)

$$\eta = M(1 - f) \quad \ldots \quad (6)$$

(1) and (3) give

$$\frac{d\xi}{\xi} = -\frac{Mdf}{1 + \theta f} \quad \ldots \quad (7)$$

Integrating under initial conditions stated above

$$\xi = \left(\frac{1 + \theta f}{1 + \theta}\right)^{-\frac{M}{\theta}} \quad \ldots \quad (8)$$

$$\therefore \zeta = \frac{(1 - f)(1 + \theta f)^{\left(1 + \frac{M}{\theta}\right)}}{(1 + \theta)^{\frac{M}{\theta}}} \quad \ldots \quad (9)$$

Now for maximum pressure $\frac{d\zeta}{df} = 0$. Denoting the values at maximum pressure by the suffix 1, we obtain from (9),

$$f_1 = \frac{M + \theta - 1}{M + 2\theta} \quad \ldots \quad (10)$$

It is impossible to reach in the gun the mathematical solution with f negative. These results have, therefore, a physical significance only if f_1 is positive or zero i.e. if $M \geqslant 1 - \theta$.

If $M < 1 - \theta$, f_1 is negative and in this case maximum pressure occurs at all-burnt, its value being given by the equation.

$$\zeta_1 = \zeta_2 = \frac{1}{(1 + \theta)^{\frac{M}{\theta}}} \quad \ldots \quad (11)$$

Using (10) we obtain from (6), (8) and (9)

$$\xi_1 = \left(\frac{M + 2\theta}{M + \theta}\right)^{\frac{M}{\theta}} \quad \ldots \quad (12)$$

$$\eta_1 = \frac{M(\theta + 1)}{M + 2\theta} \quad \ldots \quad (13)$$

$$\zeta_1 = (1 + \theta)^2 \frac{(M + \theta)^{\left(1 + \frac{M}{\theta}\right)}}{(M + 2\theta)^{\left(2 + \frac{M}{\theta}\right)}} \quad \ldots \quad (14)$$

$$= \frac{1}{\xi_1}(M + \theta)\left(\frac{\theta + 1}{M + 2\theta}\right)^2 \quad \ldots \quad (14a)$$

Also denoting the values at all-burnt by the suffix 2, we get

$$\xi_2 = (1 + \theta)^{\frac{M}{\theta}} \quad \ldots \quad (15)$$

$$\eta_2 = M \quad \ldots \quad (16)$$

$$\zeta_2 = (1 + \theta)^{-\frac{M}{\theta}} \quad \ldots \quad (17)$$

Now for a tubular charge, either by direct integration or taking limits as $\theta \to 0$ in the above results.

$$\xi = e^{M(1-f)} \quad \ldots \quad (18)$$

$$\eta = M(1-f) \quad \ldots \quad (19)$$

$$\zeta = (1-f)e^{-M(1-f)} \quad \ldots \quad (20)$$

At maximum pressure, (20) gives

$$f_1 = \frac{M-1}{M} \quad \ldots \quad (21)$$

Also

$$\xi_1 = e \quad \ldots \quad (22)$$

$$\eta_1 = 1 \quad \ldots \quad (23)$$

$$\zeta_1 = \frac{1}{Me} \quad \ldots \quad (24)$$

Again

$$\xi_2 = e^M \quad \ldots \quad (25)$$

$$\eta_2 = M \quad \ldots \quad (26)$$

$$\zeta_2 = e^{-M} \quad \ldots \quad (27)$$

Comparison of Isothermal and Non-Isothermal Models

In the following tables I, II, III and IV, we have collected for the isothermal model the values of ξ_1, η_1, ζ_1; ξ_2, η_2 and ζ_2 for $M=1$, 2, 3 and 4 in the cases when $\theta=0$, $0\cdot2$, $0\cdot5$ and $1\cdot0$. The tables V and VI give the corresponding values for the non-isothermal model, for tubular and Cord propellants respectively, calculated and collected from sources as described below :—

The values of ξ_1 and η_1 have been obtained from the formulae (Kapur[2])

$$\xi_1 = \left(1 - \frac{\eta_1 \theta^1}{1 + \theta}\right)^{-\frac{1}{\theta^1}} \quad \ldots \quad (28)$$

Where

$$\theta^1 = \frac{\theta}{M} + \tfrac{1}{2}(\gamma - 1) \quad \ldots \quad (29)$$

and

$$\eta_1 = \frac{1 + \theta}{\frac{2\theta}{M} + \gamma} \quad \ldots \quad (30)$$

For tubular propellants, the values of ζ_1 have been taken from Venkatesan[3] and for Cord Propellants from Aggarwal[4]. ξ_2, η_2 and ζ_2 have been taken from Tawakley[5] for tubular propellants and for Cord Propellants, these values have been calculated with the help of tables as given in H.M. S.O.[6]

TABLE NO. I

(*Isothermal Model*: $\theta=0$)

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	2·7183	1·0	0·3679	2·7183	1·0	0·3679
2	2·7183	1·0	0·1839	7·3891	2·0	0·1353
3	2·7183	1·0	0·1226	20·0855	3·0	0·0498
4	2·7183	1·0	0·0920	54·5980	4·0	0·0183

TABLE NO. II

(*Isothermal Model*: $\theta=0\cdot2$)

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	2·1614	0·8571	0·4079	2·4883	1·0	0·4019
2	2·3872	1·0000	0·2304	6·1917	2·0	0·1615
3	2·4828	1·0588	0·1606	15·4070	3·0	0·0649
4	2·5355	1·0909	0·1232	38·3375	4·0	0·0261

TABLE NO. III

(*Isothermal Model*: $\theta=0\cdot5$)

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	1·7778	0·750	0·4746	2·2500	1·0	0·4444
2	2·0736	1·000	0·3014	5·0625	2·0	0·1975
3	2·2282	1·125	0·2209	11·3906	3·0	0·0878
4	2·3231	1·200	0·1743	25·6289	4·0	0·0390

TABLE NO. IV

(*Isothermal Model*: $\theta=1\cdot0$)

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	1·5000	0·6667	0·5926	2·0	1·0	0·5000
2	1·7778	1·0000	0·4219	4·0	2·0	0·2500
3	1·9531	1·2000	0·3277	8·0	3·0	0·1250
4	2·0736	1·3333	0·2679	16·0	4·0	0·0625

TABLE NO. V

(*Non-Isothermal Model* : $\theta=0$)

$\gamma=1\cdot25$

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	2·3231	0·8	0·3099	2·9110	1·0	0·3006
2	2·3231	0·8	0·1550	9·9892	2·0	0·0751
3	2·3231	0·8	0·1033	42·9500	3·0	0·0147
4	2·3231	0·8	0·0775	256·0000	4·0	0·0020

TABLE NO. VI

(*Non-Isothermal Model* : $\theta=1\cdot0$)

$\gamma=1\cdot25$

M	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1	1·4589	0·6154	0·5531	2·0869	1·0	0·4193
2	1·6832	0·8889	0·3829	4·8028	2·0	0·1562
3	1·8154	1·0435	0·2929	12·6258	3·0	0·0495
4	1·9024	1·1429	0·2372	40·2567	4·0	0·0124

From the above tables we find that the effect of the neglect of the kinetic energy term is :

(1) to overestimate the maximum pressure, shot travel and velocity till the instant of maximum pressure.

(2) to underestimate the pressure and shot-travel at all burnt.

(3) not to produce any effect on the velocity at all burnt.

It is obvious, therefore, that if we want to avoid this over approximation or under approximation, we shall have to adjust the values of M or θ or of both. In the next section we study such adjustments for getting the same maximum pressure from both the models.

Adjustment of M or θ for same maximum pressure

We first consider the case in which the value $\overline{M}$ in Isothermal model corresponds to the value $\overline{M}$ in non-isothermal model for the same maximum pressure. Tables VII and VIII on the next page exhibit the relation between M and $\overline{M}$ for $\theta=0$ and $\theta=1\cdot0$ respectively.

TABLE NO. VII
(*Tubular charges*)

M for non-isothermal model	$\bar{M}$ for isothermal model
1	1·1871
2	2·3734
3	3·5613
4	4·7468

TABLE NO. VIII
(*Cord charges*)

M for non-isothermal model	$\bar{M}$ for isothermal model
1	1·1762
2	2·3539
3	3·5323
4	4·7104

We notice from these tables that to balance the effect of neglect of the kinetic energy term in the isothermal model we have to increase the value of M by about 18·7% for tubular charge and by about 17·7% for a Cord charge. It is remarkable that these percentages do not change significantly with the change in the value of M though there is a significant difference with the variation of the value of θ. We may also point out that these percentages may vary slightly with the value of γ, the percentages given above applying for the case when γ is exactly or nearly equal to 1·25.

An alternative way is to adjust the value of θ. For the isothermal model let $\bar{\theta}$ denote the value of the form—factor corresponding to the value θ for the non-isothermal model for the same maximum pressure. Then we have got the following table IX in which we have given the values of $\bar{\theta}$ for a Cord charge.

TABLE NO. IX

M	θ for Non-Isothermal Model	$\bar{\theta}$ for Isothermal Model
1	1	0·8349
2	1	0·8391
3	1	0·8403
4	1	0·8409

We find that in this case θ has to be decreased by 16·5% to 16·0% the percentages slightly decreasing with the increase of M. Here also the variation is very small.

In order to have some idea of the effect of the adjustment of M on ξ_1, η_1, ξ_2, η_2 and ζ_2 such that the maximum pressure ζ_1, remains the same for both the isothermal and non-isothermal models, we collect their values for tubular charges in table X and for Cord charges in table XI by taking adjusted value of M in the isothermal model.

TABLE NO. X.

$\overline{M}$	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1·1871	2·7183	1·0	0·3099	3·2776	1·1871	0·3051
2·3734	2·7183	1·0	0·1550	10·7337	2·3734	0·0932
3·5613	2·7183	1·0	0·1033	33·2090	3·5613	0·0284
4·7468	2·7183	1·0	0·0775	115·2150	4·7468	0·0087

TABLE NO. XI

$\overline{M}$	ξ_1	η_1	ζ_1	ξ_2	η_2	ζ_2
1·1762	1·5601	0·7406	0·5531	2·2598	1·1762	0·4425
2·3539	1·8483	1·0813	0·3829	5·1121	2·3539	0·1956
3·5323	2·0223	1·2770	0·2929	11·5702	3·5323	0·0864
4·7104	2·1385	1·4039	0·2372	26·1812	4·7104	0·3822

We re-write the equations (5) as

$$\left.\begin{aligned} \xi &= 1 + \frac{Ax}{K_o - \dfrac{c}{\delta}} \\ \eta &= \frac{AD}{FC\beta}\, v \\ \zeta &= \frac{K_o - \dfrac{c}{\delta}}{FC}\, p \\ M &= \frac{A^2 D^2}{F \beta^2 C W_1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (31)$$

Now the adjustment in M can be made by adjusting any one or more of the parameters A, $\dfrac{D}{\beta}$, F, C and W_1. Since we want p to be proportional to ζ, we shall not change K_o, C, δ and F. This leaves A, $\dfrac{D}{\beta}$ and W_1 only for adjustment. Usually the procedure is to adjust the ballistic size D but this adjustment alone will mean that the shot-travel and velocity may not be correctly predicted. Hence to get more accurate results, all the three should be so adjusted as to give as close agreement as possible. The percentages of adjustment for $\theta = o$ can be easily calculated from tables V and X, and for $\theta = 1$ from tables VI and XI.

In the next section we have plotted the $\zeta - \xi$ curves for the two models taking adjusted value of M for the isothermal model.

Pressure-Space curves with Adjusted M

The pressure for a non-isothermal model is given by the formula

$$\zeta = \frac{(1+\theta)^2}{M\,\theta^1} (\xi - 1)^{\theta^1} \xi^{(-2\theta^1 - 1)} \quad .. \quad (32)$$

where

$$\theta^1 = \frac{\theta}{M} + \frac{1}{2}(\gamma - 1) \quad .. \quad .. \quad .. \quad (33)$$

and for an isothermal model

$$\zeta = \frac{(1+\theta)^2}{\bar{M}\,\theta''} (\xi - 1)^{\theta''} \xi^{(-2\theta'' - 1)} \quad .. \quad (34)$$

where

$$\theta'' = \frac{\theta}{\bar{M}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (35)$$

The curves in the following figures (1) and (2) exhibit a relation between ζ and ξ for tubular propellants, and those in figures (3) and (4) for Cord propellants. The complete curves correspond to the non-isothermal model and the dotted curves refer to the isothermal model.

The above curves, giving the relation between ζ and ξ, give the pressure-space curves if in the adjustment of M; A, K_o, C, δ and F are not changed. If we are prepared to adjust A, the agreement between the $p - x$ cruves for the two models can be much closer.

Concluding Remarks

The isothermal model is based on three simplifying assumptions *viz.* (*a*) kinetic energy is neglected, (*b*) the covolume term is neglected and (*c*) zero shot-start pressure is assumed. It is not difficult to discuss the effects of (*b*) and (*c*). In fact this has already been done by Jain[7] and Seth[8] respectively. The most important and characteristic assumption is, however, the factor (*a*), after which the model is named. We have discussed above the effects of this assumption. It is usually stated that an adjustment by 10 to 12 % in F would compensate for this assumption. We find above that the more correct figure is 16 to 18%. To take into account the effects of all the three assumptions simultaneously would be desirable, but would require the preparation of about five hundred tables of the type given above which is, of course, beyond the resources of the present authors, but can be undertaken by a team of workers.

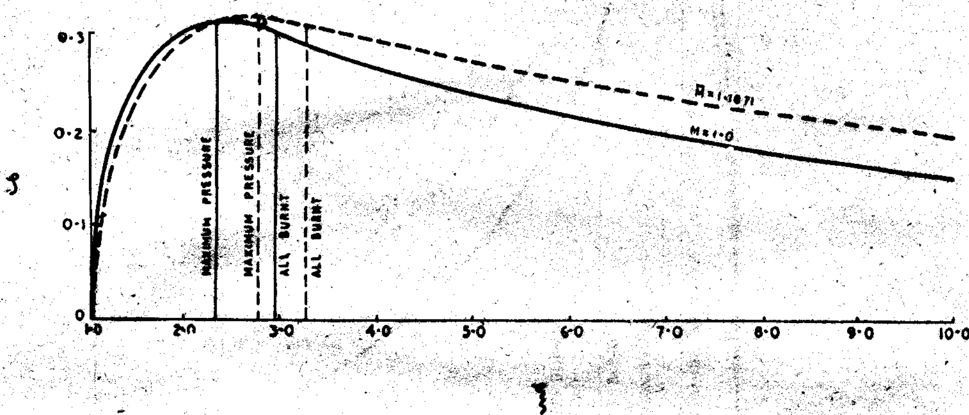
0·3
0·2
0·1
0
MAXIMUM PRESSURE
MAXIMUM PRESSURE
ALL BURNT
ALL BURNT
M=1·0
1·0
2·0
3·0
4·0
5·0
6·0
7·0
8·0
9·0
10·0

Fig. 1

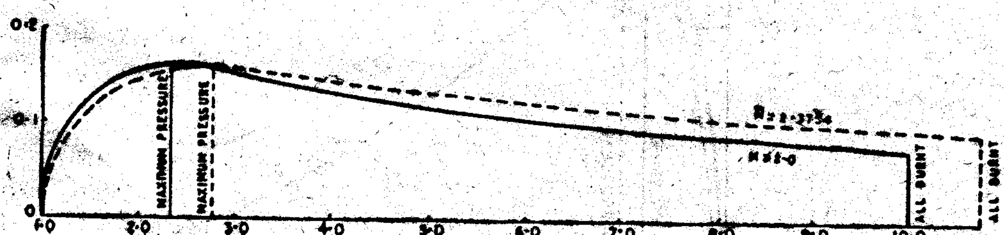
0·2
0·1
0
MAXIMUM PRESSURE
MAXIMUM PRESSURE
ALL BURNT
ALL BURNT
1·0
2·0
3·0
4·0
5·0
6·0
7·0
8·0
9·0
10·0

Fig. 2

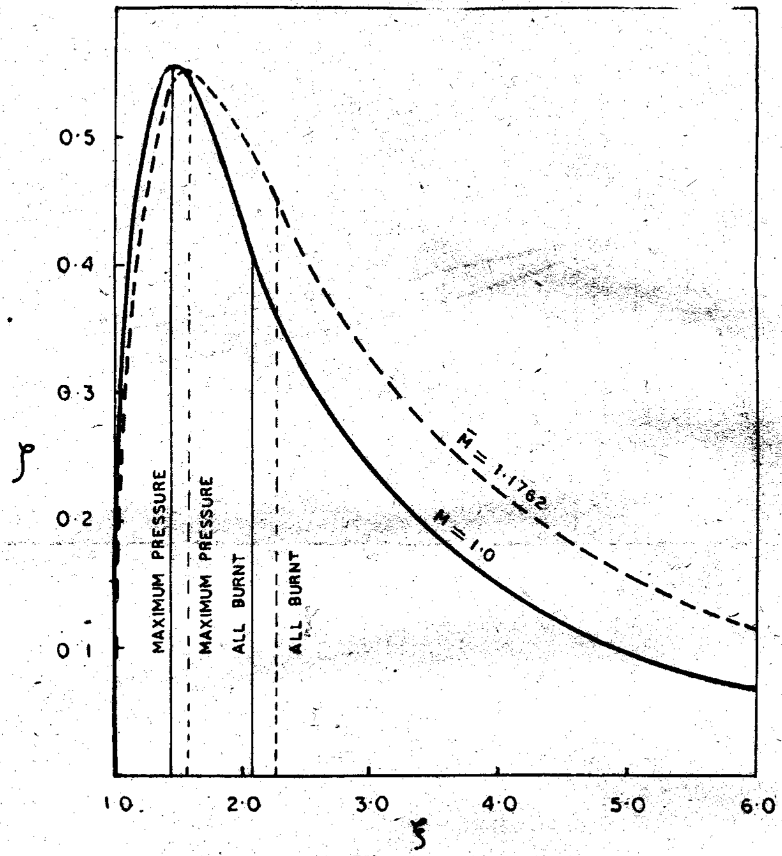
0·5
0·4
0·3
0·2
0·1
ρ
MAXIMUM PRESSURE
MAXIMUM PRESSURE
ALL BURNT
ALL BURNT
M̄ = 1·1762
M = 1·0
1·0
2·0
3·0
4·0
5·0
6·0
ξ

Fig. 3

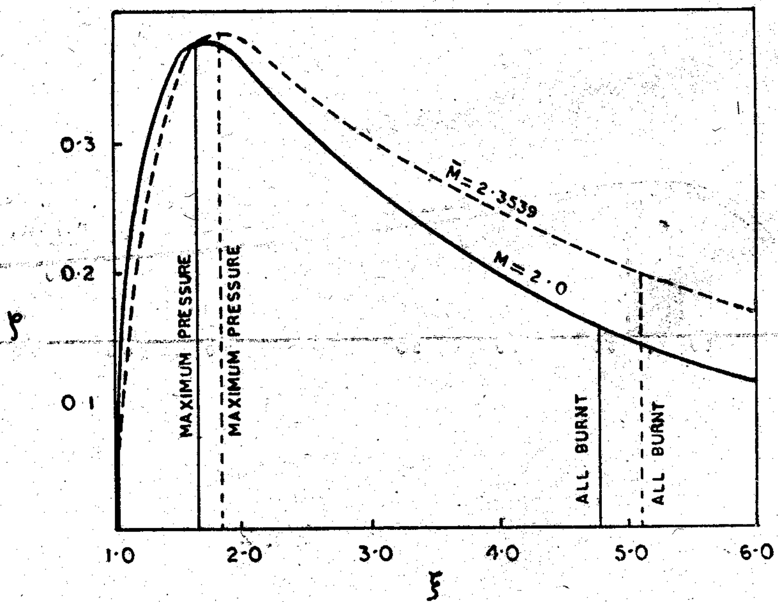
0·3
0·2
0·1
ρ
MAXIMUM PRESSURE
MAXIMUM PRESSURE
M̄ = 2·3539
M = 2·0
ALL BURNT
ALL BURNT
1·0
2·0
3·0
4·0
5·0
6·0
ξ

Fig. 4

Also in the above discussion we have adjusted either M or θ. It is obvious that if we are prepared to adjust both M and θ, the agreement between the prediction of the two models can be made still closer.

A more exhaustive discussion on these lines will form the subject matter of a later communication.

Acknowledgements

We are highly grateful to Dr. D. S. Kothari, Scientific Adviser to Defence Minister. for going through the manuscript. We are particularly thankful to Dr. R. S. Varma, F. N. I. for making some very valuable suggestions.

References

1. Corner, J., *Theory of Internal Ballistics of guns*, John Wiley and Sons, New York, 1950.
2. Kapur, J. N., Ballistics effects of shot-start pressure (to be published in *Mem. art France*).
3. Venkatesan, N. S., *Proc. Nat. Inst Sci. India*, **XVIII**, 265, 1952.
4. Aggarwal, S. P., *Proc. Nat. Inst. Sci. India*, **20**, 307, 1954.
5. Tawakley, V. B., *Def. Sci. Jour.*, **6**, 207, 1956.
6. H. M. S. O., *Internal Ballistics*, London, 1951.
7. Jain, V. K., *Def. Sci. Jour.*, **6**, 143, 1956.
8. Seth G. P., *Def. Sci. Jour.*, **6**, 173, 1956.