UNSTEADY FLOW AND HEAT TRANSFER IN A VISCOUS INCOMPRESSIBLE FLUID DUE TO THE OSCILLATIONS OF A POROUS CYLINDER WITH SUCTION

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Navier-Stokes equations which described the unsteady flow of viscous incompressible fluid a around a porous infinite circular cylinder oscillating harmonically with constant suction are integrated using Laplace transform. Curves have been drawn of the steady state part of the velocity both for suction and without suction case and it is found that for each value of suction parameter at every instant there is a concentric circle around the cylinder where there is no effect of suction, a situation which does not exist in the case of rotating cylinder. The energy equation has also been solved and the solutions are given for large and small values of time.

A number of problems of unsteady fluid flow round a rotating cylinder with and without suction have been considered earlier by various authors[1,2] under different boundary conditions. The problem of fluid motion within two concentric rotating cylinders has also been considered[3]. W. H. Schwarz[4] has studied the fluid motion due to the oscillations of an infinite circular cylinder. The problem of fluid flow and heat transfer round an oscillating porous cylinder is suggested in case of lubrication and cooling of oscillating mechanical devices and also in chemical engineering.

In this paper the problem of unsteady fluid motion and heat transfer in an incompressible fluid due to the oscillations of an infinite porous cylinder about its axis have been studied. It is assumed that there is a constant suction at the surface of the oscillating cylinder. Navier-Stokes equations describing the unsteady flow of a viscous incompressible fluid when a porous infinite circular cylinder oscillates harmonically in it have been solved using Laplace transform. It is found that in the steady flow for each value of Ω, the non-dimensional angular velocity of the oscillating cylinder, there is a concentric circle round the cylinder where there is no effect of suction, a situation which does not exist in the case of a rotating cylinder with suction at its surface. The velocity distribution curves in the fluid have been drawn for various values of Ω for the case when there is suction on the surface and the case without suction. Later the unsteady energy equation (neglecting the dissipation term) is solved for the oscillating cylinder with suction. On neglecting the dissipation term, the problem of heat transfer reduces to that of solution of energy equation in a fluid outside a porous cylinder with suction. The solution of this equation has been obtained for small and large values of time.

HYDRODYNAMIC PROBLEM AND ITS SOLUTION

In cylindrical polar coordinates, let (u_r, u_θ, u_z) be the velocity components along (r, θ, z) respectively, the axis of the cylinder coinciding with z—axis. Since the motion is rotationally symmetric and two dimensional, the derivatives with respect to θ and z are identically zero and $u_z = 0$. The equation of continuity and momentum are

$$\frac{d}{dr}(u_r\, r) = 0 \tag{1}$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^{\,2}}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) \tag{2}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r\, u_\theta}{r} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) \tag{3}$$

Equation (1) gives u_r = constant = $-S$ (say) where $S(>0)$ is the suction parameter. Substituting this value of u_r in (2) and (3), we have

$$\frac{S^2}{r^3} + \frac{u_\theta^{\,2}}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{4}$$

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\left(1+\frac{S}{\nu}\right)\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}\left(1-\frac{S}{\nu}\right)u_\theta\right] \tag{5}$$

Writing $\dfrac{S}{\nu} = 2\,(m+1)$ in equation (5) we have

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}(2m+3)\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}(2m+1)\,u_\theta\right] \tag{6}$$

Now when the infinite hollow cylinder is made to oscillate harmonically in the fluid at rest initially, the motion of the fluid may be obtained from (6) with the following boundary and initial conditions.

$$u_\theta\,(a, t) = u \sin wt,\ u_\theta\,(r, o) = 0,\ u_\theta\,(r \to \infty, t) = 0 \tag{7}$$

Using the dimensionless variables

$$V_\theta = \frac{u_\theta}{u}\ ;\ \eta = \frac{r}{a}\ ;\ T = \frac{\nu}{a^2}\,t\ ;\ \Omega = \frac{\omega a^2}{\nu} \tag{8}$$

The equations of motion and boundary conditions become

$$\frac{\partial V_\theta}{\partial T} = \frac{\partial^2 V_\theta}{\eta^2} + \frac{2m+3}{\eta}\,\frac{\partial V_\theta}{\partial \eta} + \frac{2m+1}{\eta^2}\,V_\theta \tag{9}$$

$$V_\theta\,(1, T) = \sin \Omega T,\ V_\theta\,(\eta, 0) = 0 \text{ and } V_\theta\,(\eta \to \infty, T) = 0 \tag{10}$$

Putting $V_\theta = \dfrac{V}{\eta^{m+1}}$ in (9) we get

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial \eta^2} + \frac{1}{\eta}\,\frac{\partial V}{\partial \eta} - \frac{m^2}{\eta^2}\,V \tag{11}$$

The boundary and initial conditions become

$$V\,(1, T) = \sin \Omega T\ ;\ V\,(\eta, 0) = 0 \text{ and } \underset{\eta \to \infty}{Lt}\left(\frac{V}{\eta^{m+1}}\right) \to 0 \tag{12}$$

Equations (11) with boundary condition (12) may be solved by Laplace transform method. Taking the Laplace transform of (11), we obtain

$$\eta^2\,\frac{d^2\overline{V}}{d\eta^2} + \eta\,\frac{d\overline{V}}{d\eta} - (m^2 + \eta^2 p)\,\overline{V} = 0 \tag{13}$$

This is a Bessel's equation and its solution is given by

$$\overline{V}(\eta, p) = A I_m\,(\eta p^{\frac{1}{2}}) + B K_m\,(\eta p^{\frac{1}{2}}) \tag{14}$$

where I_m and K_m are respectively the modified Bessel function of first and second kind of order m.

Condition (12) becomes

$$\overline{V}\,(1, p) = \frac{\Omega}{\Omega^2 + p^2}\ ;\ \overline{V}\,(\eta, 0) = 0 \tag{15}$$

Applying these conditions to (14), we get

$$B = 0 \quad \text{and} \quad A = \frac{-\Omega}{(\Omega^2 + p^2)\, K_m(p^{\frac{1}{2}})} \tag{16}$$

$$\therefore \quad \bar{V} = \frac{\Omega}{\Omega^2 + p^2} \; \frac{K_m(\eta p^{\frac{1}{2}})}{K_m(p^{\frac{1}{2}})} \tag{17}$$

Applying the Laplace inversion theorem to this function, we obtain

$$V = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} e^{pT} \left[\frac{\Omega}{\Omega^2 + p^2} \; \frac{K_m(p^{\frac{1}{2}}\eta)}{K_m(p^{\frac{1}{2}})} \right] dp \tag{18}$$

where C is a positive constant.

Now $K_m(p^{\frac{1}{2}})$ has a branch point at the origin. The contour of integration is taken as shown in Fig. 1. There are two simple poles located at $\pm\, i\Omega$.

Now it follows from Fig. 1 that

$$\frac{\Omega}{2\pi i} \left[\int_{C - iR}^{C + iR} \frac{e^{pT}}{\Omega^2 + p^2} \; \frac{K_m(\eta p^{\frac{1}{2}})}{K_m(p^{\frac{1}{2}})}\, dp + \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} \right]$$

$$= 2\pi i\, \Sigma \text{ residues at the poles } p = \pm\, i\Omega \tag{19}$$

The residues at the poles $p = \pm\, i\Omega$ are given by

$$R_1 = -\frac{e^{i\Omega T}}{4\pi} \; \frac{K_m(\eta\sqrt{i\Omega})}{K_m(\sqrt{i\Omega})} = \frac{e^{-i\Omega T}}{4\pi} \; \frac{K_m(\eta\sqrt{-i\Omega})}{K_m(\sqrt{-i\Omega})} \tag{20}$$

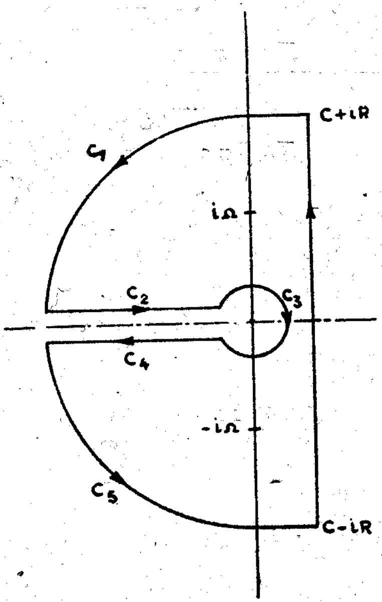

Fig. 1

Now since

$$K_m\left(Z\, i^{\pm\frac{1}{2}}\right) = e^{\pm\frac{1}{2} m\pi i} \left[Ker_m Z \pm i\, Kei_m Z \right]$$

and defining

$$Ker_m(Z) \pm i\, Kei_m Z = N_m(Z)\, e^{\pm i\phi_m(Z)}$$

and

$$\phi_m(Z) = \tan^{-1} \frac{Kei_m Z}{Ker_m Z}$$

(20) may be combined to give

$$R_1 + R_2 = \frac{1}{2\pi i} \frac{N_m(\eta\Omega^{\frac{1}{2}})}{N_m(\Omega^{\frac{1}{2}})} \sin\left[\Omega T + \phi_m(\eta\Omega^{\frac{1}{2}}) - \phi_m(\Omega^{\frac{1}{2}})\right] \tag{21}$$

Substituting these values in (19), we get

$$\left[\int_{C-iR}^{C+iR} \frac{e^{pT}\,\Omega}{2\pi i(\Omega^2+p^2)} \frac{K_m(p^{\frac{1}{2}}\eta)}{K_m(p^{\frac{1}{2}})}dp + \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5}\right]$$

$$= \frac{N_m(\eta\Omega^{\frac{1}{2}})}{N_m(\Omega^{\frac{1}{2}})} \sin\left[\Omega T + \phi_m(\eta\Omega^{\frac{1}{2}}) - \phi_m(\Omega^{\frac{1}{2}})\right] \tag{22}$$

clearly

$$\int_{C_1} \text{ and } \int_{C_5} \to 0 \text{ as } R \to \infty$$

and

$$\int_{C_3} \to 0 \text{ as } \rho \to 0$$

$$\text{On } C_2,\ p = x^2 e^{-i\pi} \text{ and on } C_4,\ p = x^2 e^{i\pi}$$

$$\therefore \quad \frac{\Omega}{2\pi i}\left[\int_{C_2} + \int_{C_4} \frac{e^{pT}}{\Omega^2+p^2} \frac{K_m(\eta p^{\frac{1}{2}})}{K_m(p^{\frac{1}{2}})}dp\right]$$

$$= \frac{\Omega}{2\pi i}\int_0^\infty \frac{2x}{\Omega^2+x^4} e^{-x^2T}\left[\frac{K_m\left(x\eta e^{\frac{i\pi}{2}}\right)}{K_m\left(x e^{\frac{i\pi}{2}}\right)} - \frac{K_m\left(x\eta e^{-\frac{i\pi}{2}}\right)}{K_m\left(x e^{-\frac{i\pi}{2}}\right)}\right]dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{x\Omega}{\Omega^2+x^4} e^{-x^2T}\left[\frac{J_m(x\eta)\,Y_m(x) - J_m(x)\,Y_m(x\eta)}{J_m^2(x) + Y_m^2(x)}\right]dx \tag{23}$$

where we have used the relation

$$K_m\left(Ze^{\pm\frac{1}{2}\pi i}\right) = \mp\tfrac{1}{2}\pi e^{\mp\frac{1}{2}\pi i(m\mp 1)}\left[J_m(Z) \mp i\,Y_m(Z)\right]$$

Therefore from (22) and (23), we have

$$V\theta = \frac{1}{\eta^{m+1}}\left[\frac{N_m(\eta\Omega^{\frac{1}{2}})}{N_m(\Omega^{\frac{1}{2}})} \sin\left[\Omega T + \phi_m(\eta\Omega^{\frac{1}{2}}) - \phi_m(\Omega^{\frac{1}{2}})\right]\right.$$

$$\left. - \frac{2}{\pi}\int_0^\infty \frac{\Omega x\, e^{-x^2T}}{\Omega^2+x^4}\left[\frac{J_m(x\eta)\,Y_m(x) - J_m(x)\,Y_m(x\eta)}{J_m^2(x) + Y_m^2(x)}\right]dx\right] \tag{24}$$

In this equation if we put $m = -1$, we get the result for suctionless case as obtained by Schwarz[4].

The graphs of the steady part of the angular velocity both for suction and suctionless cases have been drawn in Fig. 2 for $\Omega = 0{\cdot}25$, $4{\cdot}00$ and $9{\cdot}00$. From these graphs, it is found that at every instant there is a concentric circle for each Ω at which there is no effect of suction and the velocity at that circle in both the cases is the same.

The physical explanation for existence of such an instantaneous circle for every value of suction parameter, can possibly be searched out in the quasi-steady part of the expression on the right hand side of equation (24). The suction parameter (the normal velocity at the surface) interacts with the flow induced by oscillation of the porous cylinder and gives rise to the sine term. Depending upon the suction parameter, sometime during a complete oscillation this term becomes equal to the particular case of flow induced by periodic motion alone without any suction. Obviously no such situation can be imagined in the case of continuously rotating bodies where no such periodic term occurs in the solution.

THE HEAT TRANSFER PROBLEM

The energy equation giving the temperature distribution, neglecting the heat due to friction is given as

$$\rho C_T \left(\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} \right) = \lambda \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] \quad (25)$$

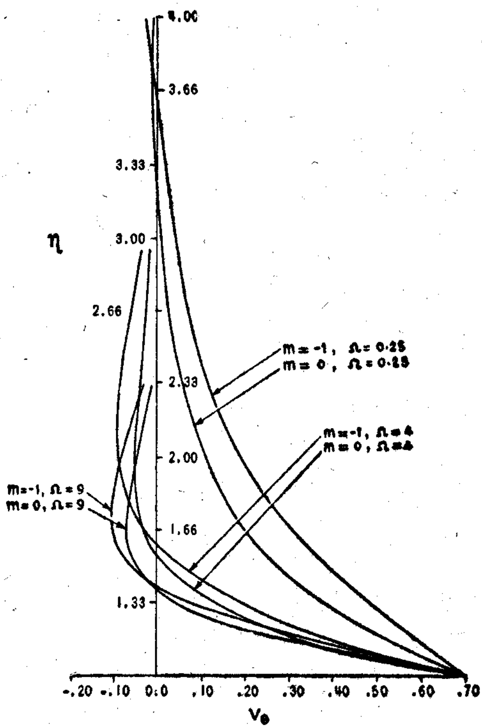

Fig. 2—Velocity distribution for constant $\Omega T = \pi/4$ and varying Ω.

The boundary conditions

$$u_r = -\frac{S}{r}, \text{ where } S \text{ is a + ive constant} \quad (26)$$

$$T(r, o) = 0 \quad \text{and} \quad T(a, t) = T_1 \quad (27)$$

Here ρ is the density, C_p is the specific heat at constant pressure and λ is the conductivity of the fluid. Putting the value of u_r given by (26) in (25), we get

$$\frac{\partial T}{\partial t} = A \frac{\partial^2 T}{\partial r^2} + \frac{B}{r} \frac{\partial T}{\partial r} \quad (28)$$

where $A = \dfrac{\lambda}{\rho C_p}$ and $B = \dfrac{\lambda}{\rho C_p} + S$

Taking Laplace transform, the equation (28) and boundary conditions (27) become

$$\frac{d^2 \bar{T}}{dr^2} + \frac{B/A}{r} \frac{d\bar{T}}{dr} - \frac{p}{A} \bar{T} = 0 \quad (29)$$

$$\overline{T}(r, 0) = 0\ ;\quad \overline{T}(a, t) = \overline{T}_1 \tag{30}$$

The solution of (29) subject to boundary conditions (30) is

$$\overline{T} = \frac{\overline{T}_1}{p} \cdot \frac{r^n}{a^n} \cdot \frac{K_n(qr)}{K_n(qa)} \tag{31}$$

where $$n = \tfrac{1}{2}\left(1 - \frac{B}{A}\right) \text{ and } q^2 = \frac{p}{A}$$

For small values of time, using the asymptotic expressions of the Bessel functions in (31), we obtain.

$$\overline{T} = \frac{T_1\, r^{n-\frac{1}{2}}}{a^{n-\frac{1}{2}}}\; e^{-q(r-a)} \left\{ 1 + \frac{(4n^2-1)(a-r)}{8qar} + \frac{(4n^2-1)}{128q^2a^2r^2}\left[(4n^2-9)\,a^2 - 2(4n^2-1)\,ar + (4n^2+7)\,r^2\right] + \ldots \right\} \tag{32}$$

Applying Laplace inversion term by term, we get

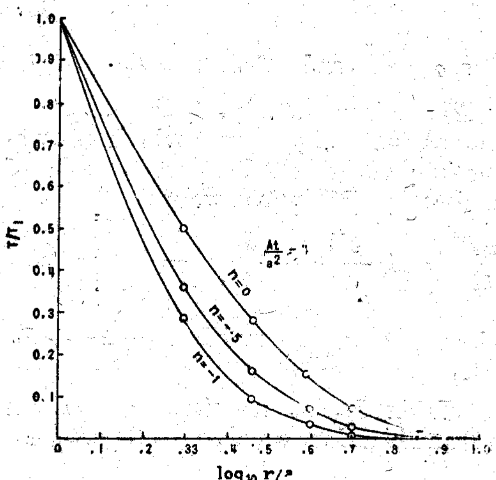

Fig. 3—Temperature in the region bounded internally by a cylinder $r = a$ with zero initial temperature and constant surface temperature T_1. The numbers on the curves are the values of n.

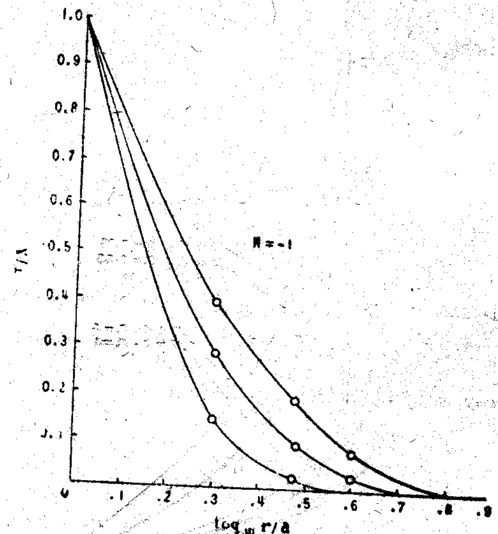

Fig. 4—Temperature in the region bounded internally by a cylinder $r = a$. The numbers on the curves are the values of At^2/a^2.

$$T = T_1 \left\{ \begin{array}{l} \left(\dfrac{r}{a}\right)^{n-\frac{1}{2}} erfc \dfrac{r-a}{2\sqrt{At}} + \dfrac{(4n^2-1)(a-r)\sqrt{At}\; r^{n-\frac{3}{2}}}{4a^{n+\frac{1}{2}}} \; i\, erfc \dfrac{r-a}{2\sqrt{At}} \\ + \dfrac{(4n^2-1)[(4n^2-9)a^2 - 2(n^2-1)ar + (4n^2+7)r^2](At)\, r^{n-\frac{5}{2}}}{32a^{n+\frac{3}{2}}} \\ \times i^2 erfc \dfrac{r-a}{2\sqrt{At}} + \cdots\cdots\cdots \end{array} \right\} \quad (33)$$

For large values of time using the technique given by Carslaw & Jaegar[5], we get

$$T = \frac{T_1 a^{2n}}{2^n \pi i r^n (n-1)!} \int_{-\theta}^{0+} e^{pt} \left\{ \begin{array}{l} \dfrac{(-1)^{n+1}}{2^{n-1} A^n} p^{n-1} \log \dfrac{pr^2c^2}{4A} \left(r^n - \dfrac{a^{2n}}{2^2 \, \lfloor n+1 \; r^n} \right) \\ + \dfrac{(-1)^n}{n! \, 2^{n-1} A^n} p^{n-1} \left(r^n - \dfrac{a^{2n}}{2^2 r^n} \right) \sum m^{-1} \\ + \dfrac{2^{n-1}(n-1)!}{r^n} p^{-1} + \dfrac{(-1)^{n+1} a^{2n} p^{n-1}}{\lfloor n+1 \; 2^{n+1} r^n A^n} \\ \times \log \dfrac{r^2}{a^2} \end{array} \right\} dp \quad (34)$$

where $\gamma = \log C$ and $\gamma = 0{\cdot}5772\ldots$ is Euler's constant.

$$\therefore\; T = T_1 \left[\frac{a^{2n}}{r^{2n}} + \frac{(-1)^{2n+1}}{2^{2n}(n+1)!} \frac{a^{2n}}{A^n} \frac{1}{t^n} \left\{ 1 - \frac{a^{2n}}{r^{2n}} \right\} + \ldots\ldots \right] \quad (35)$$

which clearly satisfies the boundary conditions at (27).

The temperature distribution for $r > a$ for various values of the dimensionless parameter At/a^2 and for various values of suction parameter obtained by use of equation (33) is depicted graphically in Fig. 3 and 4. It is clear from these figures that with the increase of the suction parameter the temperature fall is steeper along the distance from the surface of the cylinder.

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