LARGE DEFLECTION OF HEATED ORTHOTROPIC CYLINDRICAL SHALLOW SHELL

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This paper presents an analysis of large deflection of heated orthotropic shallow cylindrical shell. The governing differential equations have been derived on the basis of Berger's assumption and have been solved for a rectangular panel with simply-supported edges.

In high-speed aircraft, turbines, and machine structures, and in the fields of chemical and nuclear engineering, there are numerous problems in which thermal stresses play an important and even a primary role. It is essential to know the magnitude and effect of these thermal stresses to make a rigorous design of such components.

Although numerous references may be given for the thermal deflections and thermal stresses of elastic plates, few references may be cited for those of cylindrical shells. The author thinks that only Nowacki in his monograph[1] treated the problem of thermal deflections and thermal stresses of cylindrical and conical shells. Recently the author has published a paper on large deflection of a heated isotropic cylindrical shallow shell[2]. The purpose of this paper is to investigate the large deflection of an orthotropic cylindrical shell under uniform load and subject to a stationary temperature distribution varying through the thickness only. Governing equations have been derived on the basis of Berger's assumption[3].

FIELD EQUATIONS

We consider an elastic orthotropic shell of thickness h. We choose the lines of principal curvatures of the middle surface as x and y axes and z axis normally downwards. Let u, v, w be the components of displacement along the co-ordinate axes and let the distribution of temperature in the direction of z axis is linear, i.e.

$$T(x, y, z) = T_0(x, y) + z\,T(x, y) \tag{1}$$

where

$$\tau_0 = \frac{T_1 + T_2}{2}, \quad \tau = \frac{T_1 - T_2}{h} \tag{2}$$

and

$$T_1 = T\left(x, y, \frac{h}{2}\right), \quad T_2 = T\left(x, y, -\frac{h}{2}\right) \tag{3}$$

If K_x and K_y denote the principal curvatures at a point of the middle surface, then according to the well-known equations suggested by Karman and Tsien[4], the deformation of the middle surface pertinent to large deflection may be described by

$$\left.\begin{aligned} \bar{e}_{xx} &= \frac{\partial u}{\partial x} - K_x\,\omega + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 \\ \bar{e}_{yy} &= \frac{\partial v}{\partial y} - K_y\,\omega + \frac{1}{2}\left(\frac{\partial \omega}{\partial y}\right)^2 \\ \bar{e}_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial x}\cdot\frac{\partial \omega}{\partial y} \end{aligned}\right\} \tag{4}$$

where $\bar{e}_{xx}$, $\bar{e}_{yy}$ and $\bar{e}_{xy}$ are the components of the inplane strains assumed to be small. We assume, further, that the deflection ω is small in comparison with the radii of curvature of the middle surface of the shell.

Combining the potential energy due to bending and stretching of the middle surface of an orthotropic shell undergoing large deflection ($\bar{e}_2 = \bar{e}_{xx} \bar{e}_{yy} - \bar{e}^2_{xy}$, second strain invariant, being neglected which is Berger's assumption) with the potential energy due to heating and uniform load, q one gets the total potential energy, V, as

$$V = \frac{1}{2} \iint_S \left[D_x \left(\frac{\partial^2 \omega}{\partial x^2} \right)^2 + 2D_1 \frac{\partial^2 \omega}{\partial x^2} \cdot \frac{\partial^2 \omega}{\partial y^2} + D_y \left(\frac{\partial^2 \omega}{\partial y^2} \right)^2 + \right.$$

$$\left. + 4 D_{xy} \left(\frac{\partial^2 \omega}{\partial x \partial y} \right) + D_x \frac{12}{h^2} \bar{e}^2_1 - 2q\omega \right] dxdy -$$

$$- \iint_S \int_{-h/2}^{h/2} \left[\bar{\beta}_1 e_{xx} T \left(xyz \right) + \beta_2 e_{yy} T \left(x,y,z \right) \right] dx\, dy\, dz \tag{5}$$

where

$\bar{e}_1 = \bar{e}_{xx} + K \bar{e}_{yy}$ = first strain invariant

$$\left. \begin{aligned} \bar{e}_{xx} &= \bar{e}_{xx} - z \frac{\partial^2 \omega}{\partial x^2} \\ \bar{e}_{yy} &= \bar{e}_{yy} - z \frac{\partial^2 \omega}{\partial y^2} \end{aligned} \right\} = \text{thermal strain invariant}$$

$D_x = E'_x h^3/12, \quad D_y = E'_y h^3/12, \quad D_{xy} = Gh^3/12$

$D_1 = E'' h^3/12, H = D_1 + 2D_{xy}, K = \sqrt{D_y/D_x}$

$\beta_1 = (\alpha_2 S_{12} - \alpha_1 S_{22}) / (S_{11} S_{22} - S^2_{12}), \ \beta_2 = (\alpha_1 S_{12} - \alpha_2 S_{11})/(S_{11} S_{22} - S^2_{12})$

α_1, α_2 = co-efficients of thermal expansions $E'_x, E'_y, E'', G, S_{ij}$ = elastic constants.

Combining (1) and (5) and using the following Euler's equations of the calculus of variations to the integral, F

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0 \tag{6}$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y} \left(\frac{F}{\partial v_y} \right) = 0 \tag{7}$$

$$\frac{F}{\partial \omega} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial \ \omega} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial \omega_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial \omega_{xx}} \right) +$$

$$+ \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial \omega_{xy}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial \omega_{yy}} \right) = 0 \tag{8}$$

One gets

$$\frac{\partial}{\partial x} \left(e_1 - \frac{h^2 \beta_1 N_T}{12 D_x} \right) = 0 \tag{9}$$

$$\frac{\partial}{\partial y} \left(e_1 - \frac{h^2 \beta_2 N_T}{12 D_x K} \right) = 0 \tag{10}$$

$$\frac{\partial^2}{\partial x^2} (D_x \, \omega_{xx} + D_1 \, \omega_{yy} + \beta_1 M_T) + \frac{\partial^2}{\partial y^2} \, (D_y \, \omega_{yy} + D_1 \, \omega_{xx} + \beta_2 M_T) +$$

$$+ \frac{\partial^2}{\partial x \partial y}\left(4 D_{xy}\, \omega_{xx}\right) - \frac{\partial}{\partial x}\left(D_x \frac{12}{h^2}\, \bar{e}_1\, \omega_x - \beta_1\, N_T \omega_x\right) -$$

$$- \frac{\partial}{\partial y}\left(D_x \frac{12}{h^2}\, \bar{e}_1\, \omega_y K - \beta_2 N_T\, \omega_y\right) -$$

$$- q - \frac{12}{h^2} D_x \bar{e}_1 (K_x + K K_y) + \beta_1 N_T K_x + \beta_2 N_T K_y = 0 \qquad (11)$$

where

$$M_T = \int_{-h/2}^{h/2} zT\,(x,\, y,\, z)\, dz,\ N_T = \int_{-h/2}^{h/2} T\,(x, y, z)\, dz$$

For orthotropic materials satisfying the relation

$$\beta_1 \sqrt{D_y} = \beta_2 \sqrt{D_x}$$

one gets from (9) and (10)

$$e_1 - \frac{\beta_1 N_T h^2}{12 D_x} = e_1 - \frac{\beta_2 N_T h^2}{12\, D_x\, K} = \text{constant} = \frac{\alpha^2 h^2}{12} \qquad (12)$$

where α^2 denotes a normalised constant of integration.

Also from Eqns. (11) and (12) one gets

$$D_x \frac{\partial^4 \omega}{\partial x^4} + 2H \frac{\partial^4 \omega}{\partial x \partial y^2} + D_y \frac{\partial^4 \omega}{\partial y^4} + \beta_1 \frac{\partial^2 M_T}{\partial x^2} + \beta_2 \frac{\partial^2 M_T}{\partial y^2} -$$

$$- \alpha^2 D_x \left(\frac{\partial \omega^2}{\partial x^2} + K \frac{\partial^2 \omega}{\partial y^2}\right) + N_T \cdot (\beta_1/\rho_1 + \beta_2/\rho_2) =$$

$$= q + \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(\beta_1 N_T + \alpha^2 D_x) \qquad (13)$$

where

$$\frac{1}{\rho_1} = K_x \text{ and } \frac{1}{\rho_2} = K_y$$

METHOD OF SOLUTION

Rectangular panel simply-supported at the edges

We now approach a particular problem concerning a circular cylindrical panel simply-supported at the edges. Let the origin be located at one corner of the shell in its middle surface. Let *a, b* be the length and peripheral width of the shell, ρ_1, ρ_2, the radii as shown in Fig. 1 then we get

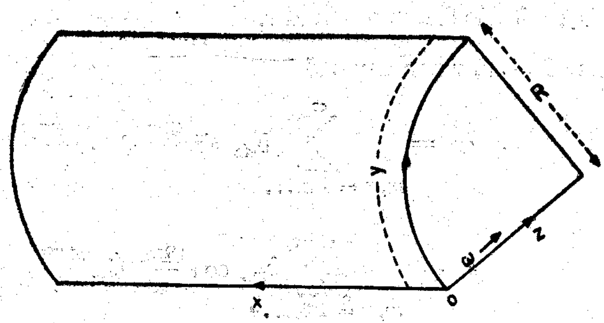

Fig. 1—Shell geometry.

Since $M_T = (T_1 - T_2)\, h^2/12 = \text{constant} = M_0$ (say) we can express it in the form of the following Fourier series

$$M_T = \sum_{m,n=1,3\ldots}^{\infty} \lambda_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{14}$$

where the Fourier co-efficient λ_{mn} is given by

$$\lambda_{mn} = \frac{16 M_0}{mn\pi^2} \tag{15}$$

The deflection ω is assumed in the form

$$\omega = \sum_{m,n=1,3\ldots}^{\infty} A_{mn} \sin \frac{m\pi n}{a} \sin \frac{n\pi y}{b} \tag{16}$$

so that this form of ω satisfies the following boundary conditions for simply-supported edges

$$\begin{aligned} \omega = 0 &= -\left(D_x \frac{\partial^2 \omega}{\partial x^2} + D_1 \frac{\partial^2 \omega}{\partial y^2}\right) + \beta_1 M_T \text{ at } x = 0,\ a \\ \omega = 0 &= -\left(D_y \frac{\partial\ \omega}{\partial y^2} + D_1 \frac{\partial^2 \omega}{\partial x^2}\right) + \beta_2 M_T \text{ at } y = 0,\ b \end{aligned} \tag{17}$$

Since q, ρ_1, ρ_2 are constants we can also express them in the form of the Fourier series

$$S_i = \sum \frac{16\, S_i}{mn\pi} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (i = 1,2,3) \tag{18}$$

where

$$S_1 = q,\ S_2 = \frac{1}{\rho_1},\quad S_3 = \frac{-1}{\rho_2} \tag{19}$$

Substituting (14), (16) and (18) into equation (13) one gets

$$A_{mn} = \frac{\frac{16q}{mn\pi} + \left(\beta_1 \frac{m^2\pi^2}{a^2} + \beta_2 \frac{n^2\pi^2}{b^2}\right)\frac{16 M_o}{mn\pi} + \beta_1 N_T \frac{16}{mn\pi\ \rho_1} + \beta_2 N_T \frac{16}{mn^2\pi\rho_2} + \frac{16}{mn\pi}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(\alpha^2 D_x + \beta_1 N_T)}{D_x \frac{m^4\pi^4}{4a^4} + 2H \frac{m^2 n^2 \pi^4}{a^2 b^2} + D_y \frac{n^4 \pi^4}{b^4} + \alpha^2 D_x \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)} \tag{20}$$

in which α^2 is still unknown.

As the inplane displacements u and v cannot be determined and we are interested only in the normal displacement w, these inplane displacements can be eliminated by integrating Eqn. (12) over the surface area of the shell by taking suitable forms of u and v.

The following forms of u and v are assumed

$$u = \sum_{m,n=1,3,\ldots}^{\infty} b_{mn} \sin \frac{m\pi_n}{a} \cos \frac{n\pi y}{b} \tag{21}$$

$$v = \sum_{m,n=1,3,\ldots}^{\infty} b_{mn} \cos \frac{m\pi_n}{a} \sin \frac{n\pi y}{b} \tag{22}$$

compatible with the boundary conditions $u=0$ at $x=0, a$; $v=0$ at $y=0, b$.
Substituting u, v and w from (21), (22) and (16) into equation (12) and integrating over the surface area of the shell one gets the following equation giving α^2

$$\frac{1}{8}\sum_{m,n=1,3,\ldots}^{\infty} A^2_{mn}\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)-2\sum_{m,n=1,3,\ldots}^{\infty}\left(\frac{K_x a}{m\pi}+\frac{K_y b}{n\pi}\right)=$$

$$=\frac{1}{12}\left(\alpha^2 h^2+\beta_1 N_T h^2/D_x\right) \tag{23}$$

Deflection $\omega(x, y)$ can now be determined from (16), (20) and (23).

Limiting case

For isotropy $D_x = D_y = H = D$ (say), $\beta_1 = \beta_2 = \frac{\alpha_t E}{1-\nu}$ $\alpha_1 = \alpha_2 = \sigma_t$ (say) where ν is Poisson's ratio and E, Young's modulus, so in the limiting case when $\alpha^2 \longrightarrow 0\ldots$ in equation (16) one gets the corresponding small thermal deflection for an isotropic rectangular plate in the form[1]

$$\omega(x,y)=\frac{16(1+\nu)\alpha_t\tau}{ab}\sum_{m,n=1,3,\ldots}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)\frac{mn\pi^2}{ab}} \tag{24}$$

Also the large deflection for isotropic shell can be deduced from (16) and (23) and is in exact agreement with the result obtain by the author[2].

NUMERICAL RESULT

For finding the deflection at a desired point one has to start from equation (23) with an assumed value of α_a and using equation (16) one gets a particular value of the temperature parameter $(T_1 - T_2)(a/h)^2\beta_1 h^3/D_x$. With this value of the temperature parameter corresponding deflection is obtained from equation (16). Variations of the non-dimensional deflection W_{max}/h for different values of temperature parameter have been plotted graphically as shown in Fig. 2 considering the following set of values.

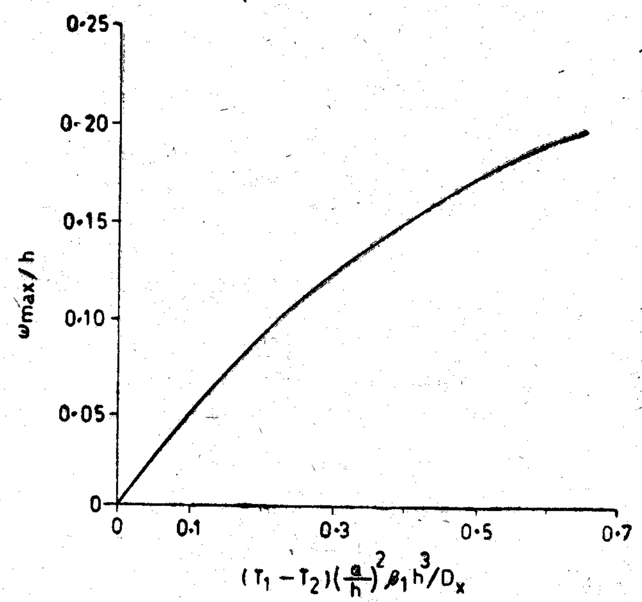

Fig. 2—Variation of central deflections with temperature parameter

$$\frac{D_y}{D_x}\ 0{\cdot}32,\ \frac{H}{D_x}=0{\cdot}2,\ a/h=20,$$

$$a/b=1;\ a/\rho_1=1,\ \frac{\beta_2}{\beta_1}=6{\cdot}5,\ \frac{\rho}{\rho_2}=1$$

$$q=0,\ a^2\beta_1 N_T/D_x=100$$

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