# THE STEADY MOTION OF A SPHERE IN A DUSTY GAS

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(*Received* 3 *May* 1972 ; *revised* 21 *July* 1972)

The paper considers the effect on the dusty gas otherwise at rest at infinity due to uniform motion of a sphere. The dust particles are assumed to have small relaxation time. Using the potential solution of gas flow at large Reynold number $R$, an equation for the concentration of dust near the sphere is derived and solved numerically. It is also shown that particles do not collide with the sphere until the Stokes number $\sigma$ is greater than 1/12 if we assume the gas flow unchanged by the presence of the dust particles.

Interest in the problem of mechanics of systems with more than one phase has developed rapidly in recent years. Situations which occur frequently are concerned with motion of a liquid or gas which contains a distribution of solid particles. Such situation occurs for example, in the movement of dust laden air, in the problems of fluidization, in the use of dust in gas cooling systems to enhance heat transfer processes, and in the process by which rain drops are formed by coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence.

Carrier[1], Rudinger[2] and Marble[3] have done extensive work on the models of dusty gas flows and shock waves in dusty gas. Later Saffman[4] formulated equations for small disturbance in plane parallel flow of a dusty gas. Following his model Michael[5] and Michael & Norrey[6] studied the steady mot on of a dusty gas past a fixed surface and arrived at approximate solutions. Here in this paper the model formulated by Saffman[4] is employed to study the motion of a sphere in a dusty gas and exact solutions are obtained. The dust is represented by a large number density $N$ of small dust particles whose volume concentration is small, but mass concentration is appreciable. It is assumed that the individual particles of dust are so small that Stokes flow approximation to their motion relative to the gas, is appreciable. The equations of motion gives rise to two additional independent parameters due to the presence of dust, *viz.* $f$, the mass concentration of the dust and $\tau$, relaxation which is representative of the time scale on which velocity of the dust adjusts itself to changes in neighbouring gas velocity. When $\tau = 0$, this adjustment is instantaneous, and we have a limiting case in which the dust moves with gas at each point. The motion in this case is closely related to flow of a clean gas. We consider here the flow of a dusty gas for small non-zero values of $\tau$ by a perturbation of the solution at $\tau = 0$. Here Reynold number is assumed to be large, and as a first step towards the solution, the paper considers in detail the perturbation of the unseparated potential flow for a sphere. The analysis shows that when a nonsingular perturbation of a potential flow is assumed, the concentration of dust particles becomes logarthmically infinite to the front stagnation point of the sphere. We find also that dust particles cannot reach the sphere except at the front stagnation point; there being a dust streamline emanating from the point which deliminates a thin dust free layer adjacent to the sphere whose thickness is of the order $\sigma a$ where $\sigma$ is Stokes number, $\tau U/\sigma$ and $U$, the velocity of sphere and $a$, its radius.

## FORMULATIONS

The equations governing the motion of dusty gas as given by Saffman[4] are :

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\,\Delta \vec{u}\right) = -\operatorname{grad} p + \mu\,\Delta^2 \vec{u} + K N\,(\vec{v} - \vec{u}) \tag{1}$$

$$\operatorname{div} \vec{u} = 0 \tag{2}$$

$$m\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\,.\,\Delta \vec{v}\right) = K\,(\vec{u} - \vec{v}) \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div}(N \vec{v}) = 0 \tag{4}$$

where $\vec{u}$ and $\vec{v}$ are velocities of gas and dust particles. $N$ is the number density of dust particles, each of mass $m$. $K$ is the Stokes coefficient of resistance, $p$, $\rho$, $\mu$, being the pressure, density and viscosity of the gas. The time relaxation parameter $\tau$ is given from (3) by $\tau = \dfrac{m}{K}$. When $\tau \to 0$ equation (3) shows that $\vec{u} \to \vec{v}$. Substituting for $\vec{u} - \vec{v}$ in (1), from (3) we have

$$\rho \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \Delta \vec{u} \right) = - \text{grad}\, p + \mu\, \Delta^2 \vec{u} - N m \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \Delta \vec{v} \right) \tag{5}$$

when $\tau \to 0$ equation (5) becomes

$$(1+f) \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \Delta \vec{u} \right) = - \frac{1}{\rho}\, \text{grad}\, p + \nu\, \Delta^2 \vec{u} \tag{6}$$

where the mass concentration of dust, $f = \dfrac{m N}{\rho}$ and $\nu = \mu/\rho$. In this limiting case when we put $\vec{u} = \vec{v}$ in (4) and using (2) we find that

$$\frac{\partial N}{\partial t} + \vec{u} \cdot \Delta N = 0$$

which means that $N$ remains constant in the neighbourhood of any given dust or gas particle. The simples case to take is one in which $N$ is uniform and equal to $N_0$ in the incident flow in which case $N = N_0$ everywhere. It then follows that $f = f_0$ a constant, in this limit. Equation (6), then represents flow of a clean gas with uniform density $\rho\,(1 + f_0)$ and viscosity $\mu$. The solution for dusty gas flow at Reynold number $R$ is then equivalent to the solution for a clean gas at the increased Reynold number $R\,(1 + f_0)$.

For the motion of a sphere with velocity $U$, the gas velocity changes on the length scale of the radius $a$ of the sphere, a perturbation on the solution for $\tau = 0$ can be obtained in terms of small dimensionless parameter $\sigma = \tau\, U/a$. For spherical dust particles of radius $d$ and density $\rho_d$, condition $\rho \leqslant 1$ becomes

$$\frac{2}{3} R \left( \frac{\rho_d}{\rho} \right) \left( \frac{d}{a} \right)^2 \leqslant 1$$

Now consider the potential flow of liquid due to the motion of the sphere in the limiting case when $\tau = 0$, neglecting for the present viscous boundary layer and separation effects. For this solution $\Delta^2 \vec{u} = 0$ in (6) and the effect of the dust is simply to scale up the pressure variations over the sphere by the factor $(1 + f_0)$.

Let $\vec{u}_0$ represent the unperturbed velocity of the dust and gas, where

$$\vec{u}_0 = \text{grad}\, \varphi$$

$$\varphi = \frac{U a^3}{2 r^2} \cos \theta \tag{7}$$

$r$, $\theta$ being spherical polar coordinates from the centre of the sphere, with $\theta = 0$ as the down stream direction and $U$ the velocity of sphere.

In the perturbation let $\vec{u} = \vec{u}_0 + \vec{u}^1$ ; $\vec{v} = \vec{u}_0 + \vec{v}^1$ represent gas and dust velocities for a small non-zero value of $\tau$, where $\vec{u}^1$ ; $\vec{v}^1$ represent small perturbation velocities of order $\tau$. Also we suppose $N = N_0 + N^1$; $f = f_0 + f^1$ and $p = p_0 + p^1$. Neglecting the internal effect of viscosity in the gas and taking only the first order terms we have from (5)

$$\vec{u_0} \cdot \Delta \vec{u^1} + \vec{u^1} \cdot \Delta \vec{u_0} + f_0 \left\{ \vec{v^1} \cdot \Delta \vec{u_0} + \vec{u_0} \cdot \Delta \vec{v^1} \right\} + f^1 \vec{u_0} \cdot \Delta \vec{u_0} = -\frac{1}{\rho} \operatorname{grad} p^1 \tag{8}$$

similarly the linearised form of (3) for the dust flow is

$$\tau \vec{u_0} \cdot \Delta \vec{u_0} = \vec{u^1} - \vec{v^1} \tag{9}$$

Equation (4) to the first order becomes

$$f_0 \operatorname{div} \vec{v^1} + \vec{u_0} \cdot \Delta f^1 = 0 \tag{10}$$

Eliminating $\vec{v^1}$ from (9) and (10) we have

$$\vec{u_0} \cdot \Delta f^1 = f_0 \tau \operatorname{div} \vec{u_0} \cdot \Delta \vec{u_0}$$

since

$$\vec{u_0} \cdot \Delta \vec{u_0} = \operatorname{grad} \frac{\vec{u_0}^2}{2}$$

the above equation can be written as

$$\vec{u_0} \cdot \Delta f^1 = f_0 \tau \Delta^2 \frac{\vec{u_0}^2}{2} \tag{11}$$

Using (7) and Laplacian in spherical polar coordinates we have

$$\vec{u_0} \cdot \Delta f^1 = f_0 \frac{9 U^2 a^6}{2 r^8} \tau (3 \cos^2 \theta + \sin^2 \theta),$$

the right hand side being always positive, and an even function about the plane $\theta = \pi/2$; the left hand side being $|\vec{u_0}| \frac{\partial f^1}{\partial s}$ ; which shows that $f^1$ increases monotonically along a streamline and the rate of increase is symmetric about $\theta = \pi/2$ .

Writing in terms of $r$ and $\theta$, (11) becomes

$$\frac{\partial f^1}{\partial r} \cos \theta + \frac{\sin \theta}{2 r} \frac{\partial f^1}{\partial \theta} = \frac{9 f_0 \tau U a^3}{2 r^5} (3 \cos^2 \theta + \sin^2 \theta) \tag{12}$$

Solving this equation we find that

$$f^1 = \frac{9 f_0 \tau U a^3}{2 r^4} \left[ -\frac{1}{4} \cos \theta \left\{ 3 + \frac{5}{6} \sin^2 \theta + \frac{25}{24} \sin^4 \theta + \frac{26}{16} \sin^6 \theta \right\} + \frac{5}{32} \sin^8 \theta \log \tan^2 \theta/2 \right] \tag{13}$$

Equation (12) was left unsolved by Michael[5] for the dusty gas flow past a fixed sphere. The stagnation points for the motion of a sphere are given $\theta = 0$ and $\theta = \pi$. Thus $f^1$, the concentration of dust particles as the sphere moves at stagnation points, is given by

$$f^1 = \frac{27 f_0 \tau U}{8 a} : f^1 = 0 \text{ at } \theta = \pi/2$$

In the dimensionless form the unperturbed streamlines are given by

$$r = c \sin^2 \theta \tag{14}$$

and (11) can be written as

$$u_0 \frac{\partial f^1}{\partial s} = f_0 \tau \Delta^2 \frac{u_0^2}{2}.$$

where $\frac{\partial f^1}{\partial s}$ represents the rate of change of $f^1$ with length along a streamline. Using (14) to eliminate $\theta$ we may deduce the following expressions for $\frac{\partial \bar{f}}{\partial r}$ and $\frac{\partial \bar{f}}{\partial \theta}$ on the streamline $K$, where $\bar{f} = \frac{2 a f^1}{9 f_0 \tau U}$

$$\frac{\partial \bar{f}}{\partial r} = \pm \frac{\left(3 - \frac{2r}{c}\right)}{r^5 (1 - r/3)^{\frac{1}{2}}} \; ; \quad \frac{\partial \bar{f}}{\partial \theta} = \frac{c^{\frac{1}{2}}}{2} \frac{\left(3 - \frac{r}{3}\right)}{r^{\frac{9}{2}}} \tag{15}$$

In the expression for $\frac{\partial \bar{f}}{\partial r}$ the —ve sign is taken for $\theta = \pi$ to $\theta = \pi/2$ and + ve sign from $\theta = \pi/2$ to $\theta = 0$.

SMALL VALUES OF $f$

If $f_0$ is small, is it follows from (10) that $f^1$ is small of the second order and to the first order (8) tells us that $\vec{u}^1 = p^1 = 0$ and equation (9) gives

$$\vec{v}^1 = - \tau (\vec{u}_0 \cdot \Delta \vec{u}_0)$$

so that

$$\vec{v} = \vec{u}_0 + \vec{v}^1$$

$$= \text{grad} \left\{ \varphi - \frac{\tau}{2} (\text{grad } \varphi)^2 \right\}$$

Equation (16) shows that $\vec{v}$ remains a potential field in this case with potential

$$\phi = U a \left[ \frac{\cos\theta}{2 r^2} - \frac{\sigma}{2 r^6} \left( \cos^2\theta + \frac{\sin^2\theta}{4} \right) \right] \tag{16}$$

The equation for dust streamlines is given by

$$\frac{dr}{r\, d\theta} = \frac{-\cos\theta + \frac{3\sigma}{r^4}\left(\cos^2\theta + \frac{\sin^2\theta}{4}\right)}{-\frac{\sin\theta}{2} + \frac{3\sigma}{r^4}(\sin\theta\cos\theta)} \tag{17}$$

It is interesting to observe that when dust particles are clean that is when $\sigma = 0$, its streamlines coincide with those of fluid particles. It is interesting to trace the divergence of the gas particles from the path given by (14). In order to do so we write the equation of the streamline in the form

$$\frac{\sin^2\theta}{r} = c + \bar{c}(\theta) \tag{18}$$

where $\bar{c}$ is a small change in $c$ of order $\sigma$, representing the displacement of the particles at an angle $\theta$; we then have

$$\frac{d\bar{c}}{d\theta} = - \frac{\sin\theta}{r^2} \frac{dr}{d\theta} + \frac{2\sin\theta\cos\theta}{r} \tag{19}$$

Eliminating $\frac{dr}{d\theta}$ from (17) and (19) we have

$$\frac{d\bar{c}}{d\theta} = \frac{6\sigma}{r^5} \left( \frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{4} \right) \sin\theta \tag{20}$$

This shows that dust path lines coincide with gas path line at a far away distance from the sphere. Since at $\theta = 0$, we have $\frac{d\bar{c}}{d\theta} = 0$, it follows that path line of dust and gas particles coincide along the direction $\theta = 0$. Integrating (20) along the streamline $r = \frac{\sin^2\theta}{c}$, we have

$$\frac{2\bar{c}}{3\sigma} = \left[ \frac{3\cot\theta}{4} \left\{ \frac{\operatorname{cosec}^7\theta}{3} + \frac{\operatorname{cosec}^5\theta}{6} + \frac{5\operatorname{cosec}^3\theta}{24} + \frac{5\operatorname{cosec}\theta}{16} \right\} + \frac{15}{128}\log\tan^2\theta/2 \right] \quad (21)$$

## DUST SEPARATION STREAMLINE

Ruling out the case in which the sphere acts as a steady source of dust, we must conclude that there is a separating streamline for the dust which starts at the first stagnation point. In the first approximation the position of this separation line will be given by the equation

$$\vec{v} \,.\, \vec{n} = 0 \quad (22)$$

Assuming $f_0$ to be small and if we write $r = a + \sigma\, a\, \delta(\theta)$, as the equation for separation line, we have to the first order

$$\delta(\theta) = \frac{1}{3\sigma} + \frac{c}{\sin^6\theta} - \frac{2}{\sin^6\theta}\left[\frac{\cos^3\theta}{3} - \frac{2\cos^5\theta}{5} + \frac{\cos^7\theta}{7}\right] - \frac{1}{2\sin^6\theta}\left[\cos\theta - \cos^7\theta - \right.$$

$$\left. - \cos^3\theta + \frac{3}{5}\cos^5\theta \right] \quad (23)$$

## CRITICAL VALUE OF $\sigma$

Although the main discussion of this paper is based on small values of $\sigma$, it is worthwhile to digress a little in order to make note of the critical value of $\sigma$ at which particles begin to collide with the sphere. This can be done on the assumption that the gas velocity is unchanged by the dust and that head-on collisions with sphere by the particles on the upstream axis will be first to occur.

The equation of motion for a particle on this axis in dimensionless form is

$$\frac{dv}{dr} = \frac{v - \frac{1}{r^3}}{\sigma v} \quad (24)$$

We have to solve this equation with boundary condition $v = 0$ at $r = \alpha$. Let us investigate the behaviour of solution of (22) at the Ist stagnation point $r = 1$. Writing $r = 1 + h$ where $h$ is small (22), becomes

$$\frac{dv}{dh} = -\left\{\frac{v - (1 - 3h)}{\sigma v}\right\} \quad (25)$$

This may be written in parametric form with parameter proportional to the time

$$\frac{dv}{dt} = 1 - 3h - v \quad (26)$$

$$\frac{dh}{dt} = \sigma v$$

Thus $v$ and $h$ have the form $e^{\lambda t}$
where
$$\lambda^2 + \lambda + 3\sigma = 0$$
when $\sigma \leqslant 1/12$ the roots $\lambda_1$ and $\lambda_2$ are real and —ve and the time taken for particles to come to stagnation point approaches infinitely like log $h$ as $h \to 0$. When $\sigma > 1/12$ we find $v$, non-zero at $h = 0$ and the particles collide with the sphere in a finite time. This result agrees with that of Michael[5].

## REFERENCES


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