BENDING OF AELOTROPIC BLOCKS—I

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(*Received* 7 *December* 1970; *revised* 17 *March* 1971)

Recently, Green & Adkins examined finite flexure of an aelotropic cuboid and indicated the conditions under which the problem may be regarded as solved. In this paper, the problem of bending of an aelotropic circular block into an ellipsoidal shell has been examined on the lines of Green & Adkins and a solution has been obtained in terms of a completely general strain energy function for both compressible and incompressible materials. The problem of bending of a circular block into a spherical shell has been obtained as a particular case.

The theory of finite deformation has received fresh impetus when Rivlin achieved one of the major advances of this century by obtaining a number of exact solutions, specially for incompressible bodies, in terms of an arbitrary strain energy function. References to various developments are found in surveys by Rivlin[1], Truesdell[2–4], Green & Zerna[5], Green & Adkins[6] and Eringen[7,8]. Though the theory has been applied to various types of problems, very few attempts[9–13] have been made to solve the problem of circular blocks bent into various shells.

Recently Green & Adkins[14] examined the finite flexure of an aelotropic cuboid and indicated the conditions under which the problem may be regarded as solved in terms of a general strain energy function. In this paper, an attempt has been made, on the lines of Green & Adkins[14] to solve the problem of bending of a circular block into an ellipsoidal shell. The solution has been obtained in terms of a completely general strain energy function for both compressible and incompressible materials. The problem of bending of a circular block into a spherical shell has been obtained as a particular case.

NOTATION AND FORMULAE

We adopt the notation and formulae of Green & Adkins[14]. The strain energy W of a homogeneous aelotropic body is expressed as a polynomial

$$W = W(e_{ij}) \tag{1}$$

in the components of strain e_{ij}. The stress tensor T^{ij} for a compressible body is given by

$$T^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}}\right)\frac{\partial \theta^i}{\partial x^r}\,\frac{\partial \theta^j}{\partial x^s}\,, \tag{2}$$

where

$$I_3 = |2e_{rs} + \delta_{rs}|\,. \tag{3}$$

For an incompressible body, $I_3 = 1$, and

$$T^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}}\right)\frac{\partial \theta^i}{\partial x^r}\,\frac{\partial \theta^j}{\partial x^s} + p\,G^{ij}\,. \tag{4}$$

The equations of equilibrium, in the absence of body forces, are

$$T^{ij}\,\|_j = 0\,. \tag{5}$$

BENDING OF AN AELOTROPIC COMPRESSIBLE CIRCULAR BLOCK INTO AN ELLIPSOIDAL SHELL

Suppose that a circular block in the undeformed state is bounded by the planes $x_3 = a_1$, $x_3 = a_2$ $(a_2 > a_1)$ and the cylinder $x_1^2 + x_2^2 = a^2$. The block is bent about the x_3-axis symmetrically into part of an ellipsoidal shell whose inner and outer boundaries are the ellipsoides of revolution obtained by revolving the confocal ellipses

$$x_3 = c \cosh \xi \cos \eta, x_1 = c \sinh \xi \sin, \eta, \xi = \xi_i, i = 1, 2 \tag{6}$$

about the x_3-axis and the edge $\eta = \alpha$. Let the y_i-axes coincide with the x_i-axes, and the curvilinear coordinates θ^i in the deformed state be a system of orthogonal curvilinear coordinates (ξ, η, φ), where φ is the angle between $y_1 y_3$ plane and the plane through the point in space and the y_3-axis. Then

$$y_1 = c \sinh \xi \sin \eta \cos \varphi, \; y_2 = c \sinh \xi \sin \eta \sin \varphi, \; y_3 = c \cosh \xi \cos \eta. \tag{7}$$

Since the deformation is symmetric about the x_3-axis, we see that

(*i*) the planes x_3 = constant in the undeformed state become the ellipsoidal surfaces ξ = constant in the deformed state;

(*ii*) the surfaces $x_1^2 + x_2^2$ = constant in the undeformed state become the surfaces η = constant in the deformed state;

(*iii*) $\tan^{-1} \dfrac{x_2}{x_1} = \varphi$.

Thus the deformation is given by

$$\xi = f(x_3), \qquad \eta = F(x_1^2 + x_2^2), \qquad \varphi = \tan^{-1} \frac{x_2}{x_1}. \tag{8}$$

The strain components are given by

$$\left.\begin{aligned}
2e_{11} &= 4x_1^2 F'^2 c^2 (\cosh^2 \xi - \cos^2 \eta) + \frac{x_2^2 c^2 \sinh^2 \xi \sin^2 \eta}{(x_1^2 + x_2^2)^2} - 1, \\
2e_{22} &= 4x_2^2 F'^2 c^2 (\cosh^2 \xi - \cos^2 \eta) + \frac{x_1^2 c^2 \sinh^2 \xi \sin^2 \eta}{(x_1^2 + x_2^2)^2} - 1, \\
2e_{33} &= f'^2 (x_3) \; c^2 (\cosh^2 \xi - \cos^2 \eta) - 1, \\
2e_{12} &= 4 x_1 x_2 F'^2 c^2 (\cosh^2 \xi - \cos^2 \eta) - \frac{x_1 x_2 c^2 \sinh^2 \xi \sin^2 \eta}{(x_1^2 + x_2^2)^2}, \\
e_{23} &= e_{13} = 0
\end{aligned}\right\} \tag{9}$$

The stress tensor (2) has components

$$\left.\begin{aligned}
T^{11} &= \frac{f'^2}{\sqrt{I_3}} \left(\frac{\partial W}{\partial e_{33}} \right), \\
T^{22} &= \frac{4F'^2}{\sqrt{I_3}} \left\{ x_1^2 \frac{\partial W}{\partial e_{11}} + x_2^2 \frac{\partial W}{\partial e_{22}} + x_1 x_2 \left(\frac{\partial W}{\partial e_{21}} + \frac{\partial W}{\partial e_{21}} \right) \right\}, \\
T^{33} &= \frac{1}{(x_1^2 + x_2^2)^2 \sqrt{I_3}} \left\{ x_2^2 \frac{\partial W}{\partial e_{11}} + x_1^2 \frac{\partial W}{\partial e_{22}} - x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\}, \\
T^{23} &= \frac{F'}{(x_1^2 + x_2^2) \sqrt{I_3}} \left\{ 2 x_1 x_2 \left(\frac{\partial W}{\partial e_{22}} - \frac{\partial W}{\partial e_{11}} \right) + \left(x_1^2 - x_2^2 \right) \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \\
T^{12} &= T^{13} = 0,
\end{aligned}\right\} \tag{10}$$

where
$$I_3 = 4c^6 f'^2 F'^2 \sinh^2 \xi \sin^2 \eta \, (\cosh^2 \xi - \cos^2 \eta)^2 \tag{11}$$

The metric tensor for the strained state of the body is given by

$$G_{ij} = \begin{bmatrix} c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 & 0 \\ 0 & c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 \\ 0 & 0 & c^2 \sinh^2 \xi \sin^2 \xi \end{bmatrix} \tag{12}$$

The equations of equilibrium (5) in this case reduce to

$$\left.\begin{aligned}
&\frac{\partial T''}{\partial \xi} + \frac{f'^2}{\sqrt{I_3}} \frac{\partial W}{\partial e_{33}} \left\{ 3 \coth \xi + \frac{\cosh \xi \sinh \xi}{\cosh^2 \xi - \cos^2 \eta} \right\} + \\
&+ \frac{4F'}{\sqrt{I_3}} \left\{ x_1^2 \frac{\partial W}{\partial e_{11}} + x_2^2 \frac{\partial W}{\partial e_{22}} + x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \left\{ - \frac{\cosh \xi \sinh \xi}{\cosh^2 \xi - \cos^2 \eta} \right\} + \\
&+ \frac{1}{(x_1^2 + x_2^2)^2 \sqrt{I_3}} \left\{ x_2^2 \frac{\partial W}{\partial e_{11}} + x_1^2 \frac{\partial W}{\partial e_{22}} - x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \left\{ - \frac{\sinh \xi \cosh \xi \sin^2 \eta}{\cosh^2 \xi - \cos^2 \eta} \right\} = 0, \\
&\frac{\partial T^{22}}{\partial \eta} + \frac{\partial T^{23}}{\partial \eta} + \frac{f'^2}{\sqrt{I_3}} \left(\frac{\partial W}{\partial e_{33}} \right) \left(\frac{\sin \eta \cos \eta}{\cosh^2 \xi - \cos^2 \eta} \right) + \\
&+ \frac{4F'^2}{\sqrt{I_3}} \left\{ x_1^2 \frac{\partial W}{\partial e_{11}} + x_2^2 \frac{\partial W}{\partial e_{22}} + x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \left\{ \frac{3 \sin \eta \cos \eta}{\cosh^2 \xi - \cos^2 \eta} + \coth \xi \right\} + \\
&+ \frac{1}{(x_1^2 + x_2^2)^2 \sqrt{I_3}} \left\{ x_2^2 \frac{\partial W}{\partial e_{11}} + x_1^2 \frac{\partial W}{\partial e_{22}} - x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} . \\
&\qquad\qquad . \left\{ - \frac{\sinh^2 \xi \sin \eta \cos \eta}{\cosh^2 \xi - \cos^2 \eta} \right\} = 0, \\
&\frac{\partial T^{33}}{\partial \varphi} + \frac{\partial T^{23}}{\partial \eta} + \frac{F'}{(x_1^2 + x_2^2) \sqrt{I_3}} \left\{ 2 x_1 x_2 \left(\frac{\partial W}{\partial e_{22}} - \frac{\partial W}{\partial e_{11}} \right) + \left(x_1^2 - x_2^2 \right) . \right. \\
&\qquad\qquad \left. . \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \left\{ \frac{2 \sin \eta \cos \eta}{\cosh^2 \xi - \cos^2 \eta} + \coth \xi \right\} = 0
\end{aligned}\right\} \quad (13)$$

These equations of equilibrium, as they stand, do not seem to admit a solution. However, a solution can be obtained if we assume η to be so small[12,13] that $\sin \eta$ can be replaced by η and $\eta = F(x_1^2 + x_2^2) = K(x_1^2 + x_2^2)^{\frac{1}{2}}$. Physically this implies that the maximum value of η, which is approximately equal to the ratio of the radius of the circular block in the deformed state to the semi-axis of the ellipsoidal shell in the direction of the y_3-axis, measured in radians, is a small quantity. This in turn implies that the deformed shell is shallow, i.e. the deflection is small. Then the equations corresponding to (8) to (12) reduce to

$$\xi = f(x_3),\ \eta = K\,(x_1^2 + x_2^2)^{1/2},\ x_1 = \frac{\eta \cos \varphi}{K},\ x_2 = \frac{\eta \sin \varphi}{K}. \quad (14)$$

$$\left.\begin{aligned}
2e_{11} &= 2e_{22} = K^2 c^2 \sinh^2 \xi - 1, \\
2e_{33} &= f'^2 c^2 \sinh^2 \xi - 1, \\
e_{12} &= e_{23} = e_{13} = 0,
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
T^{11} &= \frac{f'^2}{\sqrt{I_3}} \left(\frac{\partial W}{\partial e_{33}} \right), & T^{22} &= \frac{K^2}{\sqrt{I_3}} \left(\frac{\partial W}{\partial e_{11}} \right), \\
T^{33} &= \frac{K^2}{\eta^2 \sqrt{I_3}} \left(\frac{\partial W}{\partial e_{11}} \right), & T^{12} &= T^{23} = T^{13} = 0,
\end{aligned}\right\} \quad (16)$$

where

$$I_3 = K^4 c^6 f'^2 \sinh^6 \xi. \quad (17)$$

$$G_{ij} = \begin{bmatrix} c^2 \sinh^2 \xi & 0 & 0 \\ 0 & c^2 \sinh^2 \xi & 0 \\ 0 & 0 & c^2 \eta^2 \sinh^2 \xi \end{bmatrix} \quad (18)$$

Then the second and the third equations of (13) are satisfied identically. The first equation reduces to

$$f'^2 c^2 \sinh^2 \xi \frac{\partial}{\partial \xi}\left(\frac{\partial W}{\partial e_{33}}\right) + f'' c^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right) + + f'^2 c^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{33}}\right) - 2K^2 c^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{11}}\right) = 0 \quad (19)$$

Now

$$\frac{\partial W}{\partial \xi} = \frac{\partial W}{\partial e_{11}} \frac{\partial e_{11}}{\partial \xi} + \frac{\partial W}{\partial e_{22}} \frac{\partial e_{22}}{\partial \xi} + \frac{\partial W}{\partial e_{33}} \frac{\partial e_{33}}{\partial \xi}$$
$$= f'' c^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right) + f'^2 c^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{33}}\right) + + 2K^2 c^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{11}}\right) . \quad (20)$$

From (19) and (20) we get

$$\frac{\partial W}{\partial \xi} = \frac{\partial}{\partial \xi} \left\{ c^2 f'^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right) \right\}, \quad (21)$$

which on integration gives

$$W = c^2 f'^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right) - W_0 \quad (22)$$

where W_0 is a constant.

This gives

$$f'^2 = \frac{W + W_0}{c^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right)} . \quad (23)$$

The physical components of stress, in view of (16) to (18) are given by

$$\left.\begin{aligned} \sigma_{11} &= \frac{1}{K^2 c^2 \sinh^2 \xi} \sqrt{\left(W + W_0\right) \frac{\partial W}{\partial e_{33}}} \\ \sigma_{22} = \sigma_{33} &= \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ \left(W + W_0\right)} \\ \sigma_{12} = \sigma_{23} &= \sigma_{13} = 0 . \end{aligned}\right\} \quad (24)$$

BOUNDARY CONDITIONS

If $-R_i$, $i = 1, 2$, are the applied normal tractions on the inner and outer surfaces of the shell, we have

$$\sigma_{11} = -R_i \text{ when } \xi = \xi_i,\ i = 1, 2, \quad (25)$$

which on ubstitution from (24) give

$$\frac{1}{K^4 c^4 \sinh^4 \xi_i} \left\{ W (\xi_i) + W_0 \right\} \left(\frac{\partial W}{\partial e_{33}}\right)_{\xi=\xi_i} = R_i^2 , \quad i = 1, 2 .$$

Solving these we get the values of the constants W_0 and K.

On the edge $\eta = \alpha$ the distribution of tractions per unit arc between φ and $\varphi + d\varphi$ give rise to a force F_1 and a couple M_1 about the origin given by

$$F_1 = \alpha \int_{\xi_1}^{\xi_2} \sigma_{22}\, c^2 \sinh^2 \xi \, d\xi, \quad M_1 = \alpha \int_{\xi_1}^{\xi_2} \sigma_{22}\, c^2 \sinh^2 \xi \,(c \cosh \xi)\, d\xi. \tag{26}$$

Substituting (24) in these equations we get

$$\left.\begin{aligned} F_1 &= \alpha \int_{\xi_1}^{\xi_2} c^2 \sinh^2 \xi \left(\frac{\partial W}{\partial e_{11}}\right) \sqrt{\left(\frac{\partial W}{\partial e_{33}}\right) \Big/ \left(W + W_0\right)}\, d\xi, \\ M_1 &= \alpha \int_{\xi_1}^{\xi_2} c^2 \sinh^2 \xi \,(c \cosh \xi) \left(\frac{\partial W}{\partial e_{11}}\right) \sqrt{\left(\frac{\partial W}{\partial e_{33}}\right) \Big/ \left(W + W_0\right)}\, d\xi. \end{aligned}\right\} \tag{27}$$

These results rest on the assumption that the strain energy function W exists. When it is specified, the integrals in (27) can be evaluated. Then by writing $\xi_2 = \xi_1 + h$, where h is small, the results for thin shells (13) can be derived.

Thus to bend an aelotropic circular block into part of an ellipsoidal shell, we require a resultant force F_1 and a couple M_1 on the edge, together with prescribed normal tractions R_1 and R_2 on the inner and outer surfaces respectively.

Particular case I—Bending of an aelotropic compressible circular block into a spherical shell.

If $c \cosh \xi = c \sinh \xi$ in (6) we get the case of a circular block bent into a spherical shell, so that

$$\xi \to \infty,\ c \to 0, \text{ and } c \cosh \xi,\ c \sinh \xi \to r, \tag{28}$$

and consequently the orthogonal curvilinear coordinates (ξ, η, φ) are replaced by the spherical polar coordinates (r, θ, φ). Then the equations corresponding to (7), (8), (16), (17), (21), (24) and (27) become

$$y_1 = r \sin \theta \cos \varphi,\ y_2 = r \sin \theta \sin \varphi,\ y_3 = r \cos \theta, \tag{29}$$

$$r = f(x_3),\ \theta = K (x_1^2 + x_2^2)^{1/2},\ \varphi = \tan^{-1} \frac{x_2}{x_1}, \tag{30}$$

$$\left.\begin{aligned} T^{11} &= (f'^2/\sqrt{I_3}) \frac{\partial W}{\partial e_{33}} \\ T^{22} &= (K^2/\sqrt{I_3}) \frac{\partial W}{\partial e_{11}} \\ T^{33} &= (K^2/\eta^2 \sqrt{I_3}) \frac{\partial W}{\partial e_{11}} \\ T^{12} &= T^{23} = T^{13} = 0 \end{aligned}\right\} \tag{31}$$

$$I_3 = K^4 r^4 f'^2, \tag{32}$$

$$\frac{\partial W}{\partial r} = \frac{\partial}{\partial r} \left(f'^2 \frac{\partial W}{\partial e_{33}} \right). \tag{33}$$

$$\left.\begin{aligned} \sigma_{11} &= \frac{\sqrt{\left\{ W(r) + W_0 \right\} \dfrac{\partial W}{\partial e_{33}}}}{K^2 r^2} \\ \sigma_{22} = \sigma_{33} &= \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ \left\{ W(r) + W_0 \right\}} \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} F_1 &= \alpha \int_{r_1}^{r_2} r \left(\frac{\partial W}{\partial e_{11}}\right) \sqrt{\left(\frac{\partial W}{\partial e_{33}}\right) \Big/ \left\{W(r) + W_0\right\}}\; dr, \\ M_1 &= \alpha \int_{r_1}^{r_2} r^2 \left(\frac{\partial W}{\partial e_{11}}\right) \sqrt{\left(\frac{\partial W}{\partial e_{33}}\right) \Big/ \left\{W(r) + W_0\right\}}\; dr \end{aligned}\right\} \tag{35}$$

Particular case II—Bending of an aelotropic incompressible circular block into an ellipsoidal shell.

In this case $I_3 = 1$. Then from (17) we have

$$\frac{df}{dx_3} = \frac{1}{K^2 c^3 \sinh^3 f} \tag{36}$$

which on integration gives

$$x_3 = K^2 c^3 \left\{\frac{\cosh^3 f}{3} - \cosh f\right\} + B, \tag{37}$$

where B is an arbitrary constant.

As the internal and the external boundaries of the ellipsoidal shell are given by $\xi = \xi_i$ ($i = 1, 2$) respectively which were initially the planes $x_3 = a_1$ and $x_3 = a_2$, (37) gives

$$\left.\begin{aligned} a_1 &= K^2 c^3 \left\{\frac{\cosh^3 \xi_1}{3} - \cosh \xi_1\right\} + B, \\ a_2 &= K^2 c^3 \left\{\frac{\cosh^3 \xi_2}{3} - \cosh \xi_2\right\} + B. \end{aligned}\right\} \tag{38}$$

Solving these we get the values of the constants K and B.

Then the components of the stress tensor (4) are given by

$$\left.\begin{aligned} T^{11} &= f'^2 \frac{\partial W}{\partial e_{33}} + \frac{p}{c^2 \sinh^2 \xi}, \\ T^{22} &= K^2 \frac{\partial W}{\partial e_{33}} + \frac{p}{c^2 \sinh^2 \xi}, \\ T^{33} &= \frac{K^2}{\eta^2} \frac{\partial W}{\partial e_{33}} + \frac{p}{\eta^2 c^2 \sinh^2 \xi}, \end{aligned}\right\} \tag{39}$$

The equations of equilibrium in this case give

$$\frac{\partial p}{\partial \xi} + c^2 \sinh^2 \xi \frac{\partial}{\partial \xi} \left(f'^2 \frac{\partial W}{\partial e_{33}}\right) + 4 f'^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{33}}\right) - $$
$$- 2 K^2 c^2 \sinh \xi \cosh \xi \left(\frac{\partial W}{\partial e_{11}}\right) = 0, \tag{40}$$

$$\frac{\partial p}{\partial \eta} = 0, \qquad \frac{\partial p}{\partial \varphi} = 0. \tag{41}$$

The equations (41) show that p is a function of ξ only.

The equation (40) in view of (20) and (36) gives

$$\frac{\partial W}{\partial \xi} - \frac{\partial p}{\partial \xi} = \frac{\partial}{\partial \xi} \left\{\frac{1}{K^4 c^4 \sinh^4 \xi} \frac{\partial W}{\partial e_{33}}\right\}.$$

This on integration gives

$$p = W + W_0 - \frac{1}{K^4 c^4 \sinh^4 \xi} \frac{\partial W}{\partial e_{33}} . \tag{42}$$

From (36), (39) and (42), we get the physical components of stress as

$$\left.\begin{aligned} \sigma_{11} &= W + W_0 \\ \sigma_{22} = \sigma_{33} &= W + W_0 + K^2 c^2 \sinh^2 \xi \frac{\partial W}{\partial e_{11}} - \\ &\quad - \frac{1}{K^4 c^4 \sinh^4 \xi} \frac{\partial W}{\partial e_{33}} . \end{aligned}\right\} \tag{43}$$

BOUNDARY CONDITIONS

If the inner boundary of the shell $\xi = \xi_1$ is free from traction, we must have $\sigma_{11} = 0$ when $\xi = \xi_1$ which on substitution in (43) gives $W_0 = - W(\xi_1)$.

On the outer surface $\xi = \xi_2$ we have to apply a normal traction R given by

$$R = \sigma_{11}(\xi_2) = W(\xi_2) - W(\xi_1) . \tag{44}$$

On the edge $\eta = \alpha$, the distribution of tractions between φ and $\varphi + d\varphi$ give rise to a force F and a couple M about the origin which are given by

$$\left.\begin{aligned} F &= \alpha \int_{\xi_1}^{\xi_2} \left[(W + W_0) c^2 \sinh^2 \xi + K^2 c^4 \sinh^4 \xi \frac{\partial W}{\partial e_{11}} - \frac{1}{K^4 c^2 \sinh^2 \xi} \frac{\partial W}{\partial e_{33}} \right] d\xi \\ M &= \alpha \int_{\xi_1}^{\xi_2} \left[(W + W_0) c^2 \sinh^2 \xi (c \cosh \xi) + K^2 c^4 \sinh^4 \xi (c \cosh \xi) \frac{\partial W}{\partial e_{11}} - \right. \\ &\qquad \left. - \frac{c \cosh \xi}{K^4 c^2 \sinh^2 \xi} \frac{\partial W}{\partial e_{33}} \right] d\xi . \end{aligned}\right\} \tag{45}$$

Particular case III—Bending of an aelotropic incompressible circular block into a spherical shell.

If $c \cosh \xi = c \sinh \xi_i$ in (6), we get the case of a circular block bent into a spherical shell, so that

$$\xi \to \infty, \quad e \to 0, \quad \text{and} \quad c \cosh \xi, \quad c \sinh \xi \to r, \tag{46}$$

and consequently the orthogonal curvilinear coordinates (ξ, η, φ) are replaced by the spherical polar coordinates (r, θ, φ). Then the expressions corresponding to (43) to (45) are given by

$$\left.\begin{aligned} \sigma_{11} &= W + W_0 \\ \sigma_{22} = \sigma_{33} &= W + W_0 + K^2 r^2 \frac{\partial W}{\partial e_{11}} - \frac{1}{K^4 r^4} \frac{\partial W}{\partial e_{33}} , \end{aligned}\right\} \tag{47}$$

$$R = W(r_2) - W(r_1) \tag{48}$$

$$\left.\begin{aligned} F &= \alpha \left[\frac{(W + W_0)(r_2^2 - r_1^2)}{2} \int_{r_1}^{r_2} K^2 r^2 \left(\frac{\partial W}{\partial e_{11}} - \frac{1}{K^6 r^6} \frac{\partial W}{\partial e_{33}} \right) r\, dr \right] , \\ M &= \alpha \left[\frac{(W + W_0)(r_2^3 - r_1^3)}{2} \int_{r_1}^{r_2} K^2 r^2 \left(\frac{\partial W}{\partial e_{11}} - \frac{1}{K^6 r^6} \frac{\partial W}{\partial e_{33}} \right) r^2\, dr \right] . \end{aligned}\right\} \tag{49}$$

Thus to bend an incompressible circular block into an ellipsoidal or spherical shell, we require a resultant force F and a couple M on the edge together with a normal force R on the outer surface.

CONCLUSION

The problem of bending of an aelotropic compressible circular block bent into an ellipsoidal shell has been considered and solution has been obtained in terms of a general strain energy function. The incompressible case and the case of a block bent into a spherical shell have been obtained as particular cases.

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