OPERATIONAL BEHAVIOUR OF A COMPLEX SYSTEM UNDER PRIORITY REPAIR ECHELONS

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A mathematical model is set up to evaluate the reliability of a complex system comprising two subsystems (with standby redundancy in one subsystem). The failure and repair of units for both the subsystems follow exponential and general time distributions respectively. The switching over device for standby subsystem is not perfect and its repair is opportunistic. The repair for both subsystems is carried out under priority. The concept of waiting time for the repair of failed units in standby subsystem has also been introduced. Supplementary variable and Laplace transform techniques have been applied to obtain the transient state probabilities for such a system. From these pointwise availability has been evaluated. In the end, a particular case when repair follows exponential time distribution has been derived and asymptotic behaviour of such a system has also been examined.

In complex systems[1—3] with many components operating in series, the system reliability can be increased by identifying critical components and supporting standbys for them. The subsystem A in this problem consists of one such critical component and is thus provided with a standby unit.

In this paper, a complex system comprising two identical A-units in standby redundancy forming one sub-system A and M non- identical B-units called subsystem B has been considered. A and B in turn are connected in series. The B-units are connected in such a way that failure of either unit of this subsystem causes the system to work in reduced efficiency. The failure and repair rate of units for both the subsystems follow exponential and general repair time distributions respectively.

In subsystem A, there is a sensing and switching over device which observes the failed unit and switches to next standby unit. This device is imperfect and its repair rate follows exponential distribution with mean R^{-1}. Also the repair of failed switching over device is opportunistic which simply means that the failed A-unit (if any) is also repaired along with the repair of switching over device. Before the repair of subsystem A starts, the system may wait (with constant rate) for repair. Such situations may arise in many practical situations due to various factors like non availability of spare units or preoccupation of repair facility, etc. The repair of units for both the subsystems is carried out under two different priority repair disciplines, viz Head-of-line and preemptive resume[4,5].

ASSUMPTIONS

(*i*) The system waits for repair only when both A units fail irrespective of the state of subsystem B.

(*ii*) At any time, no more than one B-unit can fail.

(*iii*) When the jth B-unit fails, the standby unit of subsystem A may also fail.

(*iv*) The system stops working when the switching over device fails.

(*v*) The failure of both A-units brings the system into 'Down' state and then only the repair of both A units is carried out.

(*vi*) When the system is in waiting state, no repair of jth B-unit is undertaken.

NOTATIONS

M	number of B-units
j	subscript, denotes the serial number of one B-unit, $j = 1, 2, 3, \ldots\ldots M$.

λ_A	constant failure rate of independent and identically distributed A-units ; the standby failure rate is zero.
λ_j	constant failure rate of jth B-unit when no other B-unit is failed.
α	waiting time to repair for subsystem A and follows exponential time distribution.
$\eta_A(x)$, $S_A(x)$ and $\eta_j(y)$, $S_j(y)$	transient rate and probability density function; repair of both A-units and jth B-unit is completed in time x and y, respectively.
—	superbar, implies Laplace transform w.r. to t.
$\sum$	sum over j from 1 to M ; otherwise mentioned.
$\int$	denotes definite integral from 0 to ∞; otherwise mentioned.
$'$	prime denotes ordinary derivative.
(δ, μ)	state of the system; δ is the number of A-units which are failed ($\delta = 0, 1$); μ is the serial number of one failed B-unit; for $\mu = j$: $\mu = 0$ denotes no failed B-unit.
(ϕ, σ)	state of the system ; ϕ denotes the system waiting for repair of subsystem A : σ is the serial number of one failed B-unit : for $\sigma = j$: $\sigma = 0$ denotes no failed B-unit.
$p'_x(\cdot)$	denotes the probability density w.r. to x.
ϵ	pr ['successful operation of switching over device'].
$P_{\delta,0}(t)$	pr ['system is in $(\delta, 0)$ state' t].
$P_{\delta,\mu}(y, t)$	pd_y ['system is in (δ, μ) state and is under repair: elapsed repair time is y' t)
$P_{\phi,\sigma}(t)$	pr ['system is in (ϕ, σ) state' t]
$P_R(x, t)$	pd_x ['subsystem A is under repair and elapsed repair time is x' t].
$P_{r_{A,j}}(y, t)$	pd_y ['both A-units along with jth B-unit have failed ; system is under repair and elapsed repair time is y' t]
$P_{\beta\tau}(t)$	pr ['system is in down state due to the failure of switching over device; whereas one A-unit has already failed; $\tau = j$ indicates that jth B-unit has failed, $\tau = 0$ implies no failure of B-unit' t].

It is evident that,

$$P_{\delta,\mu}(t) = \int P_{\delta,\mu}(y, t)\, dy$$

$$P_R(t) = \int P_R(x, t)\, dx$$

The relations $\eta_A(x)$, $\eta_j(y)$ and $S_A(x)$, $S_j(y)$ are given by

$$S_A(x) = \eta_A(x) \exp\left[-\int_0^x \eta_A(x)\, dx\right]$$

$$S_j(y) = \eta_j(y) \exp\left[-\int_0^y \eta_j(y)\, dy\right]$$

From elementary probability considerations and continuity arguments, the following difference differential equations have been obtained for the stochastic process which is discrete in space and continuous in time.

Equations are for either model except as explicitly mentioned.

$$\left\{\frac{\partial}{\partial t}+\lambda_A+\lambda_B\right\} P_{0,0}(t)=\sum\int P_{0,j}(y,t)\,\eta_j(y)\,dy+\int P_R(x,t)\,\eta_A(x)\,dx+ + P_{\beta_0}(t)\,R \quad (1)$$

$$\left\{\frac{\partial}{\partial t}+\lambda_A+\lambda_B\right\} P_{1,0}(t)=\sum\int P_{1,j}(y,t)\,\eta_j(y)\,dy+P_{0,0}(t)\,\lambda_A\,\epsilon \quad (2)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda_A+\eta_j(y)\right\} P_{0,j}(y,t)=0 \quad \text{for model I}$$

$$=\int P_{r_A,j}(x,y,t)\,\eta_A(x)\,dx \quad \text{for model II} \quad (3)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda_A+\eta_j(y)\right\} P_{1,j}(y,t)=P_{0,j}(y,t)\,\lambda_A\,\epsilon \quad \text{for model I}$$

$$=0 \quad \text{for model II} \quad (4)$$

$$\left[\frac{\partial}{\partial t}+\alpha\right] P_{\phi,0}(t)=P_{1,0}(t)\,\lambda_A \quad (5)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\eta_A(x)\right] P_R(x,t)=0 \quad (6)$$

$$\left[\frac{\partial}{\partial t}+\alpha\right] P_{\phi,j}(t)=P_{1,j}(t)\,\lambda_A \quad (7)$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\eta_j(y)\right\} P_{r_A,j}(y,t)=0 \quad \text{for model I}$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\eta_A(x)\right\} P_{r_A,j}(x,y,t)=0 \quad \text{for model II} \quad (8)$$

$$\left[\frac{\partial}{\partial t}+R\right] P_{\beta_0}(t)=P_{0,0}(t)\,(1-\epsilon)\,\lambda_A \quad (9)$$

$$\left[\frac{\partial}{\partial t}+R\right] P_{\beta j}(t)=P_{0,j}(t)\,(1-\epsilon)\,\lambda_A \quad (10)$$

The following boundary conditions will seem to hold good :

$$P_R(0,t)=\sum\int P_{r_A,j}(y,t)\,\eta_j(y)\,dy+P_{\phi,0}(t)\,\alpha \quad \text{for model I}$$

$$=P_{\phi,0}(t)\,\alpha \quad \text{for model II} \quad (11)$$

$$P_{r_A,j}(0,t)=P_{\phi,j}(t)\,\alpha \quad \text{for model I}$$

$$P_{r_A,j}(0,y,t)=P_{\phi,j}(t)\,\alpha \quad \text{for model II} \quad (12)$$

$$P_{0,j}(0,t)=P_{0,0}(t)\,\lambda_j+P_{\beta j}(t)\,R \quad (13)$$

$$P_{1,j}(0,t)=P_{1,0}(t)\,\lambda_j \quad \text{for model I}$$

$$=P_{\ ,0}(t)\,\lambda_j+P_{0,j}(y,t)\,\lambda_A\,\epsilon\,dy \quad \text{for model II} \quad (14)$$

It has been assumed here that initially the system is operating in the state of normal efficiency i.e. $P_{0,0}(0)=1$, so that other state probabilities are zero.

SOLUTION FOR MODEL

Special Notations

$$D_j(s) = \left[1 - \bar{S}_j(s+\lambda_A)\right]\left[s+\lambda_A\right]^{-1}$$

$$C_j(s) = \left[s+\lambda_A\right]^{-1}\left[D_j(s) + \bar{S}'_j(s+\lambda_A)\right]$$

$$g_\phi(s) = \left[1 - \bar{S}_\phi(s)\right]s^{-1} \quad \phi \text{ subscript, either } A \text{ or } j.$$

$$B(s) = \left[s + \lambda_A + \lambda_B - \sum \lambda_j \bar{S}_j(s+\lambda_A) H_j(s) - K(s) R\right]$$

$$B'(0) = \left[1 - \sum \lambda_j S'_j(\lambda_A) H_j(0) - \sum \lambda_j S_j(\lambda_A) H'_j(0) - K'(0) R\right]$$

$$m(s) = \left[\bar{S}_A(s) C(s) \alpha \lambda_A \epsilon \sum \lambda_j \bar{S}_j(s) C_j(s) H_j(s)\right]$$

$$m'(0) = \alpha \lambda_A \epsilon \left[\left\{C'(0) - M_A C(0)\right\}\left\{\sum \lambda_j C_j(0) H_j(0)\right\} + \right.$$

$$\left. + C(0)\left\{\sum \lambda_j \left\{C'_j(0) H_j(0) + C_j(0) H'_j(0) - M_j C_j(0) H_j(0)\right\}\right\}\right]$$

$$G_j(s) = D_j(s)\left[1 - D_j(s) K(s) R\right]^{-1}$$

$$H_j(s) = \left[1 - K(s) R G_j(s)\right]$$

$$K(s) = \lambda_A (1-\epsilon)(s+R)^{-1}$$

$$C(s) = \lambda_A (s+\alpha)^{-1}$$

$$C'(0) = \lambda_A (\alpha^{-1})^2$$

Taking Laplace transforms of differential equations and using boundary, initial conditions and we get

$$\bar{P}_{0,0}(s) = \left[B(s) - m(s)\right] A^{-1}(s) \tag{15}$$

$$\bar{P}_{1,0}(s) = \lambda_A \epsilon \left[1 - \sum \lambda_j \bar{S}'_j(s+\lambda_A) H_j(s)\right] A^{-1}(s) \tag{16}$$

$$\bar{P}_{1,j}(s) = \lambda_j G_j(s) \bar{P}_{0,0}(s) \tag{17}$$

$$\bar{P}_{1,j}(s) = \lambda_j \left[C_j(s) \lambda_A \epsilon H_j(s) \bar{P}_{0,0}(s) + D_j(s) \bar{P}_{1,0}(s)\right] \tag{18}$$

$$\bar{P}_{\phi,0}(s) = C(s) \bar{P}_{1,0}(s) \tag{19}$$

$$\bar{P}_{\phi,j}(s) = C(s) \bar{P}_{1,j}(s) \tag{20}$$

$$\bar{P}_R(s) = g_A(s)\, C(s)\, \alpha \Big[\Big\{ \lambda_A\, \epsilon \sum \lambda_j\, \bar{S}_j(s)\, C_j(s)\, H_j(s)\, \bar{P}_{0,0}(s) \Big] + + \Big\{ 1 + \sum \lambda_j\, \bar{S}_j(s)\, D_j(s) \Big\}\, \bar{P}_{1,0}(s) \Big] \tag{21}$$

$$\bar{P}_{\beta_0}(s) = K(s)\, \bar{P}_{0,0}(s) \tag{22}$$

$$\bar{P}_{rA,j}(s) = g_j(s)\, C(s)\, \alpha\, \bar{P}_{1,j}(s) \tag{23}$$

$$\bar{P}_{\beta j}(s) = K(s)\, \bar{P}_{0,j}(s) \tag{24}$$

where

$$A(s) = \Big[\Big\{ B(s) - m(s) \Big\} \Big\{ s + \lambda_A + \lambda_B - \sum \lambda_j\, \bar{S}_j(s + \lambda_A) \Big\} \Big] - - \Big[\epsilon\, \lambda_A\, \bar{S}_A(s)\, C(s)\, \alpha \Big\{ \sum \lambda_j\, \bar{S}_j(s)\, D_j(s) + 1 \Big\} \cdot \cdot \Big\{ 1 - \sum \lambda_j\, \bar{S}'_j(s + \lambda_A)\, H_j(s) \Big\} \Big].$$

The Laplace transform of the probability that the system is in the operable state $\bar{P}_{\text{up}}(s)$ and in the failed state $\bar{P}_{\text{Down}}(s)$ can be written as

$$\bar{P}_{\text{up}}(s) = \bar{P}_{0,0}(s) + \bar{P}_{1,0}(s) + \sum \Big[\bar{P}_{0,j}(s) + \bar{P}_{1,j}(s) \Big]$$
$$= \Big[\Big\{ 1 + \sum \lambda_j\, G_j(s) + \lambda_A\, \epsilon \sum \lambda_j\, C_j(s)\, H_j(s) \Big\} \Big\{ B(s) - m(s) \Big\} + + \Big\{ 1 + \sum \lambda_j D_j(s) \Big\} \Big\{ \lambda_A\, \epsilon \Big\{ 1 - \sum \lambda_j\, \bar{S}'_j(s + \lambda_A)\, H_j(s) \Big\} \Big]\, A^{-1}(s)$$

$$\bar{P}_{\text{Down}}(s) = \bar{P}_{\phi,0}(s) + \bar{P}_R(s) + \bar{P}_{\beta_0}(s) + \sum \Big[\bar{P}_{rA,j}(s) + \bar{P}_{\beta j}(s) + \bar{P}_{\phi,j}(s) \Big]$$
$$= \Big[K(s) + \Big\{ g_A(s) \sum \bar{S}_j(s)\, C(s)\, \alpha + C(s) + g_j(s)\, C(s)\, \alpha \Big\} \cdot \cdot \Big\{ \sum \lambda_j\, C_j(s)\, \lambda_A\, \epsilon\, H_j(s) + K(s) \sum \lambda_j\, G_j(s) \Big]\, \bar{P}_{0,0}(s) + + \Big[g_A(s)\, C(s)\, \alpha + C(s) + \Big\{ g_A(s) \sum \bar{S}_j(s)\, C(s)\, \alpha + + C(s) + g_j(s)\, C(s)\, \alpha \Big\} \sum \lambda_j\, D_j(s) \Big]\, \bar{P}_{1,0}(s)$$

It is interesting to note that

$\bar{P}_{\text{up}}(s) + \bar{P}_{\text{Down}}(s) = s^{-1}$ which ought to be

ASYMPTOTIC BEHAVIOUR

The asymptotic behaviour can be derived with the help of Abels' Lemma i.e.,

$$\mathrm{Lim}_{s\to 0}\left[s\,\bar{f}(s)\right] = \mathrm{Lim}_{t\to\infty}\left[tf(t)\right] = f \text{ (say)}$$

provided the limit on the right exist. The result obtained are straight forward and can be given as under.

$$P_{\mathrm{up}} = \Big[\Big\{ 1 + \sum \lambda_j G_j(0) + \lambda_A \epsilon \sum \lambda_j C_j(0) H_j(0) \Big\} \Big\{ B(0) - m(0) \Big\} +$$

$$+ \Big\{ 1 + \sum \lambda_j D_j(0) \Big\} \lambda_A \epsilon \Big\{ 1 - \sum \lambda_j S'_j(\lambda_A) H_j(0) \Big\} \Big] H^{-1}$$

$$P_{\mathrm{Down}} = \Big[K(0) + \Big\{ M_A C(0)\alpha + C(0)\ \alpha\ M_j \Big\} \Big\{ \sum \lambda_j C_j(0) \lambda_A \epsilon H_j(0) +$$

$$+ K(0) \sum \lambda_j G_j(0) \Big\} P_{0,0} + \Big[M_A C(0)\alpha + C(0) + M_A C(0)\alpha +$$

$$+ C(0) + M_j C(0) \alpha \sum \lambda_j D_j(0) \Big] P_{1,0}$$

where

$$P_{0,0} = \Big[B(0) - m(0) \Big] H^{-1}$$

$$P_{1,0} = \lambda_A \epsilon \Big[1 - \sum \lambda_j S'_j(\lambda_A) H_j(0) \Big] H^{-1}$$

and

$$H = A'(0)$$

Here M_A and M_j are the mean time to repair of two failed A-units and jth B-unit respectively and can be defined as,

$$M_A = \int x\, S_A(x)\, dx$$

$$M_j = \int y\, S_j(y)\, dy$$

SOLUTION FOR MODEL II

Special notations (also see the special notations for model I)

$$R(s) = \Big[s + \lambda_A + \lambda_B - \sum \lambda_j \bar{S}_j(s + \lambda_A) \Big]$$

$$R'(0) = \Big[1 - \sum \lambda_j S'_j(\lambda_A) \Big]$$

$$T(s) = \Big[\sum \lambda_j \bar{S}_j(s + \lambda_A) D_j(s) H_j(s) + 1 \Big] \lambda_A$$

$$T'(0) = \lambda_A \epsilon \left[\sum \lambda_j \left\{ S'_j(\lambda_A) D_j(0) H_j(0) + S_j(\lambda_A) D'_j(0) H_j(0) + \right.\right.$$
$$\left.\left. + S_j(\lambda_A) D_j(0) H'_j(0) \right\} \right]$$

The method applied for the solution of model II is the same as in model I. The solution is given from equation (25) to (34)

$$\bar{P}_{0,0}(s) = R(s)/A(s) \tag{25}$$
$$\bar{P}_{1,0}(s) = T(s)/A(s) \tag{26}$$
$$\bar{P}_{0,j}(s) = \lambda_j D_j(s) H_j(s) \bar{P}_{0,0}(s) \tag{27}$$
$$\bar{P}_{1,j}(s) = \lambda_j D_j(s) \left[\bar{P}_{1,0}(s) + \lambda_A \epsilon D_j(s) H_j(s) \bar{P}_{0,0}(s) \right] \tag{28}$$
$$\bar{P}_{\phi,0}(s) = C(s) \bar{P}_{1,0}(s) \tag{29}$$
$$\bar{P}_R(s) = g_A(s) C(s) \alpha \bar{P}_{1,0}(s) \tag{30}$$
$$\bar{P}_{\phi,j}(s) = C(s) \bar{P}_{1,j}(s) \tag{31}$$
$$\bar{P}_{rA,j}(s) = g_A(s) C(s) \alpha \bar{P}_{1,j}(s) \tag{32}$$
$$\bar{P}_{\beta 0}(s) = K(s) \bar{P}_{0,0}(s) \tag{33}$$
$$\bar{P}_{\beta j}(s) = K(s) \bar{P}_{0j}(s) \tag{34}$$

where

$$A(s) = \left[B(s) R(s) - \bar{S}_A(s) C(s) \alpha T(s) \right]$$

$\bar{P}_{up}(s)$ and $\bar{P}_{Down}(s)$ can also be obtained in the same way as in model I.

$$\bar{P}_{up}(s) = \left[1 + \sum \lambda_j D_j(s) H_j(s) + \lambda_A \epsilon \sum \lambda_j D_j^2(s) H_j(s) \right] \bar{P}_{0,0}(s) +$$
$$+ \left[1 + \sum \lambda_j D_j(s) \right] \bar{P}_{1,0}(s).$$

$$\bar{P}_{Down}(s) = \left[K(s) + \sum \lambda_j D_j(s) H_j(s) + C(s) \left\{ 1 + g_A(s) \alpha \right\} \sum \lambda_j D_j^2(s) \right.$$
$$\left. . \lambda_A \epsilon H_j(s) \right] \bar{P}_{0,0}(s) + \left[C^2(s) \left\{ g_A(s) \alpha + 1 \right\} \sum \lambda_j D_j(s) \right] . \bar{P}_{1,0}(s).$$

In this case also

$$\bar{P}_{up}(s) + \bar{P}_{Down}(s) = s^{-1}$$

The asymptotic behaviour can also be derived on the same lines as suggested for model I. The results are given as under

$$\bar{P}_{up} = \left[1 + \sum \lambda_j D_j(0) H_j(0) + \lambda_A \epsilon \sum \lambda_j D_j^2(0) H_j(0) \right] P_{0,0} +$$
$$+ \left[1 + \sum \lambda_j D_j(0) \right] P_{1,0}$$

$$P_{Down} = \left[K(0) + \sum \lambda_j D_j(0) H_j(0) + C(0) (1 + M_A \alpha) \sum \lambda_j D_j^2(0) \lambda_A . \right.$$
$$\left. . \epsilon H_j(0) \right] P_{0,0} + \left[C^2(0) (M_A \alpha + 1) \sum \lambda_j D_j(0) \right] P_{1,0}$$

where

$$P_{0,0} = \left[\lambda_A + \lambda_B - \sum \lambda_j S_j(\lambda_A) \right] H^{-1}$$

$$P_{1,0} = \left[\sum \lambda_j S_j(\lambda_A) D_j(0) H_j(0) + 1 \right] H^{-1}$$

$$H^{-1} = \left[B'(0) R(0) + B(0) R'(0) - C'(0) T(0) + M_A C(0) T(0) - C(0) T'(0) \right]$$

Particular Case (with constant repair rate)

When repair rate is constant, $\overline{S}_A(s) = \dfrac{\eta_A}{s + \eta_A}$ and $\overline{S}_j(s) = \dfrac{\eta_j}{s + \eta_j}$. These can be substituted in the equations of above model.

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