STRESSES IN A THIN ISOTROPIC ELASTIC PLATE HAVING A HYPOTROCHOIDAL HOLE UNDER UNIFORM PRESSURE

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(*Received* 31 *October* 1977)

The solution of the problem for an isotropic infinite plate with a hypotrochoidal hole under the action of all round uniform pressure has been obtained in a closed form. It is assumed that the plate is unloaded at infinity. Stress components have been found out. In particular, solutions for plates with elliptic and circular holes have been obtained.

General methods of solving two-dimensional boundary value problems by the asymptotic use of the complex variable theory are now well known[1]. It is, however, found that in many cases direct methods developed by Sen[2,3] make the solutions simpler. Making use of these methods, he has successfully solved various types of problems [4–6]. Later on these methods were followed and successfully employed by different workers in the field [7,8].

In this paper the same method has been followed and the problem of an infinite plate with a hypotrochoidal hole has been obtained in a closed form, for the case of an approximate square hole the solution agrees to that obtained by Sen[6]. In particular, solutions have been obtained for the cases of elliptic and circular holes.

FUNDAMENTAL EQUATIONS

The small deformation of an infinite plate consisting of a homogeneous isotropic elastic material with a hypotrochoidal hole has been considered.

The mapping function is

$$Z = R\,(e\zeta + c e^{-m\zeta}) \tag{1}$$

where $Z = x + iv$, $\zeta = \xi + \eta i$, $R > 0$, $c \geqslant 0$ and m is a positive integer. The boundary $\xi = 0$ gives a hypotrochoidal hole in the Z-plane. It is assumed that the boundary of the hole is under the action of all round uniform pressure of magnitude P and the plate is unloaded at infinity.

In absence of body forces the stress components satisfying the equations of equilibrium and compatibility are given as follows[6].

$$\frac{\widehat{\xi\xi}}{h^2} = \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} + \frac{4\odot}{h^2} + F \tag{2}$$

$$\frac{\widehat{\eta\eta}}{h^2} = \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} + \frac{4\odot}{h^2} - F \tag{3}$$

$$\frac{\widehat{\xi\eta}}{h^2} = \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \eta} - G \tag{4}$$

F and G are conjugate harmonic functions and $\odot$ is a plane harmonic function satisfying the equation

$$\frac{\partial^2 \odot}{\partial \xi^2} + \frac{\partial^2 \odot}{\partial \eta^2} = 0$$

Equations (2)-(4) can be rewritten for convenience in use as :

$$\frac{\widehat{\xi\xi}}{h^4} = \frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \eta}\left(\frac{\odot}{h^2}\right) - \frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \xi}\left(\frac{\odot}{h^2}\right) + \frac{4\odot}{h^4} - \\ - \odot\left\{\frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \eta}\left(\frac{1}{h^2}\right) - \frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \xi}\left(\frac{1}{h^2}\right)\right\} + \frac{F}{h^2} \tag{5}$$

$$\frac{8\widehat{\eta\eta}}{h^4} = \frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \xi}\left(\frac{\odot}{h^2}\right) - \frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \eta}\left(\frac{\odot}{h^2}\right) + \frac{4\odot}{h^4} -$$

$$- \odot\left\{\frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \xi}\left(\frac{1}{h^2}\right) - \frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \eta}\left(\frac{1}{h^2}\right)\right\} - \frac{F}{h^2} \tag{6}$$

$$\frac{8\widehat{\xi\eta}}{h^4} = - \frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{\partial}{h^2}\right) - \frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{\odot}{h^2}\right) +$$

$$+ \left\{\frac{\partial r^2}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{1}{h^2}\right) + \frac{\partial r^2}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{1}{h^2}\right)\right\} - \frac{G}{h^2} \tag{7}$$

SOLUTION OF THE PROBLEM

Stress components $\widehat{\xi\xi}$, $\widehat{\eta\eta}$, $\widehat{\xi\eta}$ are given in terms of harmonic function $\odot$ and conjugate functions F and G by (5) to (7).

Boundary conditions are

$$\widehat{\xi\xi} = -P, \quad \widehat{\xi\eta} = 0 \quad \text{at } \xi = 0 \tag{8}$$

Making use of the mapping function (1), we get

$$x = R\,(\xi \cos\eta + ce^{-m\xi}\cos m\eta)$$

$$y = R\,(e\xi \sin\eta - ce^{-m\xi}\sin m\eta)$$

Hence, $r^2 = R^2\{e^{2\xi} + 2ce^{(1-m)\xi}\cos(m+1)\eta + c^2e^{-2m\xi}\}$ (9)

and $1/h^2 = R^2\{e^{2\xi} - 2mc\,e^{(1-m)\xi}\cos(m+1)\eta + m^2c^2e^{-2m\xi}\}$ (10)

Following Sen[6], we assume,

$$\odot = B\left[1 - Re\left\{(e^{\zeta} + mce^{-m\zeta})/(e^{\zeta} - mce^{-m\zeta})\right\}\right] \tag{11}$$

where B is an unknown constant and Re denotes the real part. The second condition of (8), with the help of (9)—(11) gives

$$G/h^2 = -4BR^4 m(m+1)(1+mc^2)c\sin(m+1)\eta \text{ at } \xi = 0 \tag{12}$$

We notice that G is the imaginary of the function

$$F + iG = 4BR^4(1+mc^2)\,(e^{\zeta} + m^2c\,e^{-m\zeta})/(e^{\zeta} - mc\,e^{-m\zeta}) \tag{13}$$

The function G obtained from (13) satisfies (12). Hence, we have

$$G/h^2 = -4BR^4m\,(m+1)(1+mc^2)\,c\,e^{(1-m)\xi}\sin(m+1)\eta \tag{14}$$

and

$$F/h^2 = 4BR^4\,(1+mc^2)\left\{e^{2\xi} + m(m-1)\,c\,e^{(1-m)\xi}\cos(m+1)\eta - m^3c^2e^{-2m\xi}\right\} \tag{15}$$

The first boundary condition of (8), then gives

$$B = -2P \tag{16}$$

Determining the unknown constant B, functions F, B and $\odot$ are known and the problem is completely solved.

For $m = 3$, the result agrees to that obtained by Sen[6].

EXPRESSIONS FOR STRESSES

Stress components from (5) - (7) are part in the forms

$$\widehat{\xi\xi}/P = \left\{ mN^2 - HK - 2LT - 2L(mN^2 + HM)/T - (1 + mc^2) M \right\}/T^2 \tag{17}$$

$$\widehat{\eta\eta}/P = \left\{ HK - mN^2 - 2LT - 2L(mN^2 + HM)/T + (1 + mc^2) M \right\}/T^2 \tag{18}$$

$$\xi\eta/P = N \left\{ K + mH + 2L(M - mH)/T - m(1 + mc^2) \right\}/T^2 \tag{19}$$

where

$$H = e^{2\xi} - (m-1)\, c\, e^{(1-m)\xi} \cos(m+1)\eta - mc^2 e^{-2m\xi}$$

$$K = 2m^3 c^2 e^{-2m\xi} - m(m-1)\, c\, e^{(1-m)\xi} \cos(m+1)\eta$$

$$L = m^2 c^2 e^{-2m\xi} - {}^{m}c\, e^{(1-m)\xi} \cos(m+1)\eta$$

$$M = e^{2\xi} + m(m-1)\, c\, e^{(1-m)\xi} \cos(m+1)\eta - m^3 c^2 e^{-2m\xi}$$

$$N = (m+1)\, c\, e^{(1-m)\xi} \sin(m+1)\eta$$

and

$$T = e^{2\xi} - 2mc\, e^{(1-m)\xi} \cos(m+1)\eta + m^2 c^2 e^{-2m\xi}$$

The stress intensity factor is given by

$$S_{m,c} = [\widehat{\eta\eta}]_{\xi=0} \tag{20}$$

Particular Cases

(*I*) When $m = 1$, the boundary $\xi = 0$ corresponds to an elliptic hole in the z-plane. Corresponding expressions for stress components are obtained on putting $m = 1$ in (17) to (19). Variations of the stress intensity factor $S_{1,c}$ for $c = 0{\cdot}25$ and $c = 0{\cdot}5$ are shown in the Fig. 1.

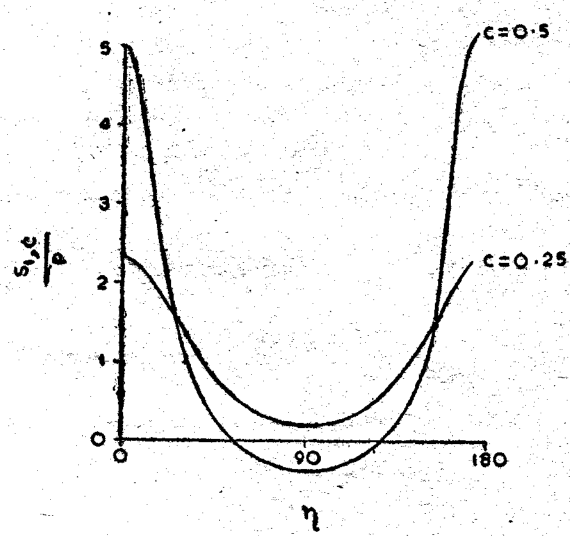

Fig. 1—Variation of stress intensity factor.

(*II*) When $c = 0$, the hole is a circle. The stress intensity factor is given by

$$S_{m,0} = P.$$

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