GEOMETRY OF STREAMLINES IN FLUID FLOW THEORY

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Intrinsic properties of lines of flow has been studied by employing anholonomic co-ordinate system consisting of s—lines which are streamlines, n—lines the involutes of s—lines and b—lines the locus of Centre of spherical curvature of s—lines. This gives rise to only two geometric parameters and interesting results have been obtained. It was also shown that velocity can be expressed in terms of geometric parameters. Constancy of velocity along binormal line implies existence of Lamb surfaces for the motion. It is found the motion is not irrotational unless it is plane motion. In generalised screw motion it is found that $wn/v=$ constant along the stream line.

The partial differential equations governing fluid flow are nonlinear and hence it is very difficult to give correct picture of fluid flow. Synge[1], Truesdell & Bjorgum[2] turned the attention of researchers to the kinematical aspects of the fluid flow, and introduced theory of surfaces and curves related to streamlines which do exist in fluid flow theory. This approach will help engineers and technologists who are faced with the problems of flow of fluids through tubes assuming the shape of space curves. It can be applied in aeronauticals engineering. Biologist can make use of it to study blood circulations.

Marris[3], Marris & Passman[4], Purushotham & Samba Shiva Rao[5] employed anholonomic co-ordinate system involving 8 parametres to study the kinematic aspect of fluid flow.

The anholonomic system employed by them consists of $\vec{s}$— the tangent line, $\vec{n}$— the normal lines that is lines whose tangent is in direction of $\vec{n}$, and $\vec{b}$— the binormal lines that is the lines whose tangent is in direction of $\vec{b}$—. The $\vec{s}$, $\vec{n}$, $\vec{b}$ lines so defined does not yield clear picture of fluid flow. Since involute and locus of centre of spherical curvature of $\vec{s}$ — lines have their tangents in the direction of $\vec{n}$ and $\vec{b}$ respectively[6] these curves defines an anholonomic co-ordinate system and thus adopted in this investigation as anholonomic co-ordinate system. As a result more interesting results are obtained in terms of only to geometric parameters related to $\vec{s}$ — lines only.

INTRINSIC RELATIONS

Considering $\vec{s}$, $\vec{n}$, $\vec{b}$ as triply unit tangent principal normal and binormal vectors along the curves of congruences formed by stream lines, involutes of stream lines and the locus of centre of spherical curvature of stream lines, denoting the lengths along $\vec{s}$, $\vec{b}$, lines as s, b and denoting curvature of stream line by k and torsion by τ, we have the following intrinsic relations[6]

$$\frac{d\vec{r}}{ds} = \vec{s} = \frac{\vec{v}}{v}\,;\quad \frac{d\vec{s}}{ds} = k\,\vec{n}\,;\quad \frac{d\vec{n}}{ds} = \tau\,\vec{b} - k\,\vec{s}\,;\quad \frac{d\vec{b}}{ds} = -\tau\,\vec{n} \tag{1}$$

$$\frac{d\vec{s}}{dn} = \frac{\vec{n}}{e-s}\,;\quad \frac{d\vec{n}}{dn} = \frac{\tau\,\vec{b} - k\,\vec{s}}{k\,(e-s)}\,;\quad \frac{d\vec{b}}{dn} = -\frac{\tau\,\vec{n}}{e-s} \tag{2}$$

$$\frac{d\vec{s}}{d_b} = \tau_2\,\vec{n}\,;\quad \frac{d\vec{n}}{db} = -\tau_2\,\vec{b} + k_2\,\vec{s}\,;\quad \frac{d\vec{b}}{db} = -k_2\,\vec{n} \tag{3}$$

Where the curvature and torsion of $\vec{n}$ — lines is given by

$$k_1 = \frac{\sqrt{k^2+\tau^2}}{k\,(e-s)}\,;\quad \tau_1 = \frac{k\,\tau' - k'\,\tau}{k\,(k^2+\tau^2)\,(e-s)}\,;\quad \frac{dn}{ds} = k\,(e-s) \tag{4}$$

The curvature and the torsion of $\vec{b}$ — lines are given by

$$k_2 = \tau \frac{ds}{db} \; ; \; \tau_2 = k \frac{ds}{db} \; ; \; \frac{db}{ds} = \frac{\rho}{\sigma} + \frac{d}{ds}\left(\sigma \rho' \right) \tag{5}$$

$$\text{Div } \vec{s} = \frac{1}{e-s} = \frac{k_1 k}{\sqrt{k^2 + \tau^2}} \; ; \; \text{Div } \vec{n} = -\left(k + \tau_2 \right) \tag{6}$$

$$\text{Div } \vec{b} = \frac{-\tau}{k\,(e-s)} \tag{7}$$

$$\text{Curl } \vec{s} = k_2 \vec{b} - \tau_2 \vec{s} \tag{8}$$

$$\text{Curl } \vec{n} = \frac{\tau \vec{s} + k \vec{b}}{k\,(e-s)} + \left(k_2 - \tau \right) \vec{n} \; ; \; \text{Curl } \vec{b} = k_2 \vec{s} - \tau \vec{b} \tag{9}$$

Using the solenoidal property on (8) and (9) we have

$$\text{Div Curl } \vec{s} = \frac{d\tau_2}{ds} + \frac{\tau + \tau_2}{e-s} - \frac{dk}{db} = 0 \tag{10}$$

$$\begin{aligned} \text{Div Curl } \vec{n} &= \frac{d}{ds}\left\{ \frac{\tau}{k\,(e-s)} - k\left(k_2 - \tau \right) \right\} + \frac{d}{dn}\left(k_2 - \tau \right) + \\ &+ \frac{d}{db}\left(\frac{1}{e-s} \right) - \tau_2 \left(k_2 - \tau \right) = 0 \end{aligned} \tag{11}$$

$$\text{Div Curl } \vec{b} = \frac{dk_2}{ds} + \frac{k_2}{e-s} + \frac{\tau_2}{k\,(e-s)} + \frac{d\tau}{db} = 0 \tag{12}$$

The values of 8 parametres, given by Marris[3] will have the following values in the new co-ordinating system.

$$\theta_{sb} = 0 \; ; \; \theta_{ns} = k \frac{ds}{dn} \; ; \; \theta_{bs} = 0; \; \theta_{bn} = \tau \frac{ds}{dn} \; ; \; \theta_{nb} = -\tau \frac{ds}{dn}$$

$$\theta_{sn} = -k \; ; \; \psi_s k \frac{ds}{db} \; ; \; \Omega_s = -k \frac{ds}{dn} = \vec{s}.\text{Curl } \vec{s} \tag{13}$$

$$\Omega_n = \vec{n}.\text{ Curl } \vec{n} = -\left(k \frac{ds}{db} + \tau \right) ; \; \Omega_b = \vec{b}.\text{ Curl } \vec{b} = -\tau \tag{14}$$

Arc distances measured along the vector lines of $\vec{s}$, $\vec{n}$ and $\vec{b}$ can never be employed as curvilinear co-ordinates. A curvilinear co-ordinate system can be constructed on these vector lines if and only if they are curves of intersection of a triply orthogonal system of surfaces. If

$$\vec{n}.\text{ Curl } \vec{n} = 0 \text{ then } \vec{n} = \psi \text{ grad } U \tag{15}$$

The vector lines $\vec{n}$ form a normal congruence. When this condition is satisfied the unit vectors $\vec{s}$ and $\vec{b}$ span the tangent plane to the surface

$$U\,(x^2) = \text{constant}, \tag{16}$$

The condition (15) is equivalent to

$$k^2 + \tau^2 + \frac{d}{ds}\left(\sigma\ \rho'\right) = 0 \tag{17}$$

In the case of study incompressible motion the conservation equation Div $\vec{v} = 0$ reduces to

$$\frac{dv}{ds} + \frac{v}{e—s} = 0 \tag{18}$$

Integrating along the streamline we have $v = v_0\ e^{-\int \frac{k_1\, k ds}{\sqrt{k^2 + \tau^2}}}$ this shows that the velocity $\vec{v}$ is a function of geometric parametres of the streamline

$$\text{Curl}\ \vec{v} = -v\ \tau_2\ \vec{s} + \frac{dv}{db}\ \vec{n} + \left(v\ \text{k} - \frac{dv}{dn}\right)\ \vec{b} \tag{19}$$

If the motion is irrotational Curl $\vec{v} = 0$ hence $\tau_2 = 0$ which implies that $\tau = 0$ hence irrotational motion cannot exist unless it is a plane motion.

Suppose in addition to the satisfying $\Omega_n = 0$ the motion is such that $w_n = \vec{n}$. Curl $\vec{v} = \frac{dv}{db} = 0$ in this case not only the Lamb's surfaces exist for the motion because vorticity vector is always tangential to the stream surface, every surface $U\ (x^2) =$ constant will be a Lamb surface.

Complex lamellar acceleration : The acceleration is complex lamellar if and only if $\vec{a}$. Curl $\vec{a} = 0$ for a study flow Curl $\vec{a} =$ Curl (Curl $\vec{v} \wedge \vec{v}$).

The condition for complex lamellar can be written as

$$\frac{dv}{ds}\left\{\frac{d}{ds}\left(\text{k}\ \frac{ds}{db}\ v\right) + \text{k}^2\ \frac{ds}{db}\ \frac{de}{dn}\ v\right\} - \text{k}\left(v\ \frac{d^2\ v}{db\,ds} - \frac{dv}{ds}\cdot\frac{dv}{db} + \Omega_n\ \text{k}\ v^2\right) = 0 \tag{20}$$

When $\Omega_n = 0$ writing a_s and a_n for tangential and normal components of acceleration we have

$$-\ \text{k}\ \frac{ds}{db} = -\frac{1}{\psi}\ \frac{\partial}{\partial b}\left(\psi\ \frac{a_n}{a_s}\right) \tag{21}$$

This equation gives the abnormality of the stream line.

Steady generalised screw motion : A steady generalised screw motion is defined by $\vec{v} = mv_B$ where $\vec{v}_B \wedge$ Curl $v_B = 0$.

It is easy to verify that this motion as a complex lamellar acceleration, the equation for complex lamellar becomes

$$-\ 2\ \frac{dv}{ds}\ k_s\ w_n + k\ \frac{d}{ds}\left(v\ w_n\right) + \Omega_n\, w_b = 0 \tag{22}$$

where

$$w_n = 0 \quad \text{we have} \quad \frac{\partial}{\partial s}\left(\frac{w_n}{v}\right) = 0$$

which shows that $\dfrac{w_n}{v}$ = constant along a stream line.

REFERENCES

1. SYNGE, J.L., *Proc. Lon. Math. Soc.* (1937), 376-416.
2. TRUESDELL & BJORGUM, *Zeicts for Angew Math. U. Mech.*, 40 (1960), 9-14.
3. MARRIS, A.W., *Archieves for Rational Mechanics and Analysis*, 35(2), (1969), 122-168.
4. MARRIS, A.W. & PASSMAN, S.L., *Archieves for Rational Mechanics and Analysis*, 32(1), (1969).
5. PURSHOTAM & RAO, P. SAMBA SHIV, *Tensors N.S. (Japan)*, 22 (1971).
6. WEATHERBURN, C.E., "Differential Geometry of Three Dimensions" (Cambridge University Press), 1964.