A METHOD TO FIND OUT FLOW VARIABLES BEHIND THE SHOCK SURFACE IN STRONG POINT EXPLOSIONS IN AXI-SYMMETRIC MEDIA

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In this paper a new method using Eulerian Co-ordinate system is developed under local radiality assumption to study the flow variables behind the shock surface in cases of strong point explosions in an axi-symmetric exponential medium and in a medium where density in the undisturbed medium obeys some axi-symmetric power laws.

Several authors have studied the cases of a strong point explosions in an exponential gaseous mediums in recent years. We have already (the method devised by Laumbach and Probstein[1]), modified B.K. theory of Sachdev[2,3], and lastly the technique devised by Oppenheim[4] *et al.* Sakashita[5] has demonstrated how Laumbach and Probstein method can be utilised to study a spheroidal model which is of great interest in astrophysical context.

In this paper a new technique is developed to study the character of flow variables behind the shock front in axi-symmetric models. This new method can help us to study the medium where density law in the undisturbed medium is a function of the product of the Eulerian distance r and the polar angle θ which can be measured from the vertical axis of symmetry.

We have presented two cases in this paper. The first part is a case where the density law in the undisturbed medium is of the form given by

$$\rho_0 = \rho_c \, e^{\beta r \Theta}$$

where ρ_c is some central density assumed to be constant, β is some constant and Θ is a function of the polar angle θ.

In the second part we deal with a medium where the density law in the undisturbed state is of the form given by

$$\rho_o = \rho_c \, (r\Theta)^{\alpha}$$

where α is some constant.

We have shown how the case studied in the first part helps one to study models similar to that chosen by Sakashita[5]. We have also shown how the method can be fitted to the case of explosion in cold exponential atmosphere similar to one studied by Laumbach and Probstein[1].

Actually, in the first part, we have extended the method developed by Deb Ray and Bhowmick[6] to study the exponential medium to the case of spheroidal symmetry or axial symmetry in an exponential medium.

In the second part we have extended the solutions obtained by Deb Ray[7] for spherically symmetric medium to the case where the undisturbed medium satisfies some axi-symmetric power law of density.

Local radiality assumption is assumed here for both the parts following Laumbach and Probstein[1] who used the same assumption in similar medium in developing their method to obtain very good results. Our method is only valid as long as the local radiality assumption is valid.

EQUATIONS OF MOTION

If we consider the general equations in Eulerian Spherical polar co-ordinate system and neglect the gradients in the θ and ϕ direction (i.e. under the assumption of local radiality) we obtain the equation of motion as

$$u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \tag{1}$$

The equation of continuity reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho u r^2) = 0 \tag{2}$$

Energy equation is

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} = \gamma \frac{p}{\rho} \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r}\right] \tag{3}$$

where

u (q_r in general equation[8])

r=particle velocity in the radial direction,

p=pressure at a distance r,

ρ=density in the disturbed fluid at a distance r,

γ=ratio of specific heats which is assumed to be constant.

So, the above equations are equations when there is a spherical symmetry in a fluid medium. But here, they are obtained from general equation considering gradients in θ and ϕ direction to be negligible.

Let us introduce a new set of variables namely $r_1 = r\Theta$ and $t_1 = t\Theta$, where Θ is a function of polar angle θ only. θ might be measured from the axis of symmetry (vertically upwards or downwards) so the equation (1), (2) & (3) can be re-written as

$$\frac{\partial u}{\partial t_1} + u \frac{\partial u}{\partial r_1} = - \frac{1}{\rho} \frac{\partial p}{\partial r_1} \tag{4}$$

$$\frac{\partial \rho}{\partial t_1} + \frac{1}{r_1^2} \frac{\partial}{\partial r_1} (\rho u r_1^2) = 0 \tag{5}$$

$$\frac{\partial p}{\partial r_1} + u \frac{\partial p}{\partial r_1} = \gamma \frac{p}{\rho} \left(\frac{\partial \rho}{\partial t_1} + u \frac{\partial \rho}{\partial r_1}\right) \tag{6}$$

So, we are now in a new (r_1, t_1) plane.

We notice that a relation of the type $\delta r = q_r \delta t$ remains valid if we replace r by $r\Theta$ and t by $t\Theta$. Therefore the particle velocity u ($= q_r$) is unaffected by such a transformation.

So, in forming the above equations only the variables r and t are changed; p and ρ are not touched and γ is constant.

Now, in both the cases we have neglected the effects due to presence of magnetic field, gravity or motion of the undisturbed medium.

Firstly we further consider the explosion to be intense so that the counter pressure may be neglected.

In the second part we have considered the cases of explosion which is strong but finite. We have taken into consideration of a pressure law in the undisturbed medium.

Density law in the undisturbed fluid can be written to be of the form given by $\rho_0 = \rho_c f(r\,\Theta)$.

In the first part $f(r_i\,\Theta) = \rho^{\beta r \Theta}$ and in the second part $f(r\Theta) = (r\Theta)^{\alpha}$

FORMULATION OF THE PROBLEM AND BOUNDARY CONDITION

Case I

Let ρ_o = density in the undisturbed medium

$$= \rho_c \, e^{\beta r} \Theta \tag{7}$$

where r denotes the Eulerian distance of a fluid particle in a definite direction ρ_c and β being suitable constants.

In the changed co-ordinates $\rho_o = \rho_c \, e^{\beta r_1}$ (8)

With the equations (4), (5) and (6), the equations governing the flow can be put in the form

$$\frac{\partial E}{\partial t_1} + \frac{1}{2r_1} \frac{\partial}{\partial r_1} (2uIr_1) = 0 \tag{9}$$

$$\frac{\partial \rho}{\partial t_1} + u \frac{\partial \rho}{\partial r_1} + \rho \frac{\partial u}{\partial r_1} + \frac{2 \rho u}{r_1} = 0 \tag{10}$$

$u, p, \rho,$ representing the velocity, pressure and density, respectively of gas at a radial distance r_1 from the centre of explosion, γ is the ratio of specific heats

$$\text{where} \quad E = \tfrac{1}{2} \rho u^2 + p/(\gamma - 1) \tag{11}$$

$$I = \tfrac{1}{2} \rho u^2 + \frac{\gamma}{\gamma - 1} \cdot p \tag{12}$$

The equation (6), (9) & (10) are same as in our earlier works[6]. Hence we arrive at the following differential equations:

Equation (6) can be put in the form

$$\frac{1}{p} \frac{\partial p}{\partial r_1'} = \frac{\gamma}{\rho} \cdot \frac{\partial \rho}{\partial r_1'} + \frac{\gamma \beta R_1}{u' - 1} \tag{13}$$

Equation (9) can be put in the form

$$\frac{dE}{dr_1'} = \frac{2}{r_1'} \frac{d}{dr_1'} (r_1'^2 \; u' \, I) \tag{14}$$

Equation (10) is of the form

$$\frac{1}{\rho} \frac{\partial \rho}{\partial r_1'} = \frac{2\lambda R_1}{u' - 1} + \frac{1}{1 - u'} \cdot \frac{\partial u'}{\partial r_1'} + \frac{1}{1 - u'} \cdot \frac{2u'}{r_1'} \tag{15}$$

where

$$\left. \begin{aligned} r_1' = \frac{r_1}{R_1} = \frac{r}{R} = r' \text{ and } u' = \frac{u}{V_1} = \frac{u}{V} \\ \lambda R_1 = - \log t_1/t_{10} = - \log t/t_0 = \beta/2 R_1 \end{aligned} \right\} \tag{16}$$

Now, the equations (13), (14), (15) with the help of (16) reduces to

$$\frac{1}{p} \frac{\partial \phi}{\partial r'} = \frac{\gamma}{\rho} \frac{\partial \rho}{\partial r'} + \frac{\gamma \cdot 2 \log t/t_0}{u' - 1} \tag{17}$$

$$\frac{dE}{dr'} = \frac{2}{r'} \cdot \frac{d}{dr'} (r'^2 u' I) \tag{18}$$

$$\frac{1}{\rho} \cdot \frac{\partial \rho}{\partial r'} = \frac{-2 \log t/t_0}{u' - 1} + \frac{1}{1-u'} \cdot \frac{\partial u'}{\partial r'} + \frac{1}{1-u'} \cdot 2u'/r' \tag{19}$$

The equations (17), (18) and (19) are nothing but the differential equations 19, 23 and 25 obtained by us in our previous work[6].

The differential equations are solved in our earlier works[6]. Therefore, the numerical solutions obtained in the work[6] are still valid here as long as the flow remains locally radial and the explosion is strong. Therefore the conclusion given at the end of the paper[6] remains valid in case of axi-symmetric flow discussed above.

Case II

Under the same set of transformation the medium where the density law is given by

$$\rho_o = \rho_c (r\Theta)^a = \rho_c r_1^a \tag{20}$$

a and ρ_c being some constants, is considered.

The equations (4), (5) and (6) are still the basic equations are to be solved. Following Deb Ray[7], we arrive at the following differential equation in (r_1, t_1) plane

Eq. (5) and (6) reduces to

$$\frac{1}{p} \cdot \frac{\partial p}{\partial r_1} - \frac{\gamma - 1}{\rho} \cdot \frac{\partial \rho}{\partial r_1} = -\frac{2}{r_1} - \frac{\left\{a(\gamma-1)+1\right\} \Big/ R_1 - \dfrac{1}{V_1} \dfrac{\partial u}{\partial r_1}}{r_1/R_1 - u/V_1} \tag{21}$$

Now, if we resubstitute $r_1\ r\ \Theta$; $V_1 = V$, $\frac{r_1}{R_1} = \frac{r}{R}$, $R_1 = R\,\Theta$ the equation (21) reduces to

$$\frac{1}{p} \cdot \frac{\partial p}{\partial r} - \frac{\gamma - 1}{\rho} \cdot \frac{\partial \rho}{\partial r} = -\frac{2}{r} - \frac{\dfrac{1}{R}\left\{a(\gamma-1)+1\right\} - \dfrac{1}{V}\dfrac{\partial u}{\partial r}}{r/R - u/V} \tag{22}$$

which is the equation corresponding to ordinary power law in spherically symmetric model.

Eqn. 5 can be put in the form

$$\frac{1}{\rho} \cdot \frac{\partial \rho}{\partial r_1'} = \frac{1}{r_1' - u_1'} \frac{\partial u_1'}{\partial r_1'} + \frac{2}{r_1' - u_1'} \cdot \frac{u_1'}{r_1'} + \frac{a}{r_1' - u_1'} \tag{23}$$

Now, using the substitutions

$$r_1' = \frac{r_1}{R_1} = \frac{r}{R} = r' \text{ and } u_1' = \frac{u}{V_1} = \frac{u}{V} = u'$$

Equation (23) reduces to

$$\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial r'}=\frac{1}{r'-u'}\frac{\partial u^1}{\partial r'}+\frac{2}{r'-u'}\cdot\frac{u'}{r'}+\frac{\alpha}{r'-u'} \tag{24}$$

which is again an equation in spherically symmetric model[7]. Thus we see that the equation (22) and (24) are same as obtained by Deb Ray[7] though density in the undisturbed medium is a function of ($r_1\Theta$) under local radiality assumption.

Therefore, the exact analytic solutions obtained by Deb Ray[7] are still valid.

CONCLUSION

What we conclude here is that a proper set of transformation is available which can transform the solution set of spherically symmetric problem to the solution of axi-symmetric problems provided the density law in the undisturbed medium is given by $\rho_o = \rho_c\,(r\Theta)^\alpha$. Therefore, we can further conclude that the solution for p/p_1, ρ/ρ_1 and u/u_1 as obtained by Deb-Ray[7] is still valid in axi-symmertic medium as long as the flow remains locally radial i.e. the gradients in θ and ϕ direction remain negligible. Exact analytic solutions obtained by Deb-Ray[7] is thus extended to the case of axi-symmetric medium. The solutions are not reproduced as we have not solved the differential equations here which are already solved in the reference mentioned.

Application of Case I

In the first case the density law is of the form $\rho_o = \rho_c\, e^{\beta r\Theta}$

If we choose $\Theta = \pm \left[1 + \frac{e_1^{\,2}}{1-e_1^{\,2}}\cos^2\theta\right]^{1/2}$ where θ is the polar angle measured from vertical axis of symmetry, we get a model of spheroidal symmetry similar to one used by S. Sakashita[5]. Our results are valid in the model as we did not put any restriction on in our derivation of equations. So we can conclude that the mass behind the shock front remains in thin shell if the explosion is strong even if the total energy is not constant. This is the basic assumption made by Sakashita[5] following Laumbach and Probstein[1]. Moreover we can obtain the flow variable behind the shock front in such a model by the method devised above.

The other interesting features is if we put $\Theta = \cos\theta$, but θ is measured from the downward direction of vertical axis, we come to the problem of explosion in cold exponential atmosphere similar to one discussed in great details by Laumbach and Probstein[1]. So the assumption of Laumbach and Probstein[1] regarding the formation of thin shell behind the front is justified if the explosion is strong and flow remains locally radial even when the energy is not constant. Moreover we can deduce the flow variable behind the shock front in a such case.

Application of Case II

Let $\Theta = \cos\theta$, then the density law in the undisturbed medium is $\rho_o = \rho_c\, h^\alpha$ where h is the height of a fluid particle from a level surface in the undisturbed medium. Hence the case become a case of strong but finite explosion in the atmosphere where density law is some power of height from a level surface. This type of density law is more real than the usual assumption of density varying as power of radial distance from the source of explosion.

CONCLUSION

We have shown in both the cases I and II that the solution obtained for flow variables behind shock front in the cases of spherical symmetries can be extended to the case of spheroidal symmetry provided the density in the undisturbed medium is a function of the product of Eulerian distance and some function of θ— the polar angle measured from the vertical axis of symmetry and the local radiality assumption is valid. Here, in case II we have shown also how the similarity method can be made useful in the case where the density in the undisturbed medium is given by a law of the form $\rho_o = \rho_c\,(r\Theta)^\alpha$ where ρ_o, ρ_c, r, Θ, α are already defined in the text. The method shown in this paper will be applicable in many other cases in future in dealing with axi-symmetric blast wave problems.

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