A SOLUTION OF THE EQUATIONS OF INTERNAL BALLISTICS OF GUNS BASED ON THE LAGRANGE DENSITY APPROXIMATION

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(*Received* 30 *August* 1976)

This paper gives the solution of the equations of internal ballistics of a gun by taking the exact form of density of the propellant gases obtained on the basis of Lagrange approximation. A general quadratic form function is used.

Methods based on the conventional density function, C/AX, have been given by many Ballisticians[1,2]. This gas density function is obtained under the assumption that whole of the charge is burnt before the movement of the shot. This assumption leads us to the conclusion that the density of the reaction products continuously decreases with the motion of the shot. However, in reality propellant is not burnt completely when shot starts moving. Due to burning of the propellant, gas will continue to evolve. Thus density of the propellant gas must increase particularly in the initial stages of the movement of the shot. It is true that when shot is moving rapidly through the gun barrel the density of the reaction product must decrease and particularly it is so after all burnt stage. So in reality density of the propellant gas should increase in the initial stage of the shot movement and then may start decreasing even before the all burnt point is reached. This is important when the ratio of the charge weight to shot weight is not very small.

To consider the gradual burning of the propellant, Chugh[3] assumed a more realistic gas density function given by (Cz/AX). Chugh[4,5], has worked out the solution of the equations of internal ballistics of a gun based on his density function. The theory has been extended for composite charge by Prasad[6,7]. Recently S. P. Aggarwal, J. K. Modi, and P. S. Varma[8,9], have given a new approximation to the density of the propellant gases viz. $\frac{Cz}{(AX - C/2\delta)}$. Some modifications in the density functions were made by adding or omitting some terms from the density functions of the propellant gases. But by taking the exact form of density function of the combustion products based on the Lagrange approximation, $\rho = \frac{Cz}{[k_0 + Ax - C(1-z)/\delta]}$, it is possible to get a solution of the equations of internal ballistics of a gun. In this paper, taking Lagrange density approximation, a method has been given to solve the internal ballistics equations. Density-distance curve obtained by this method gives that the density increases in the initial stages of the movement of the shot and finally starts decreasing as the shot motion becomes more rapid inside the gun barrel.

BASIC EQUATIONS

A brief description of the equations of the internal ballistics of a gun, for the system under consideration is given as follows

Form Functions

The form function gives a relationship between the charge mass fraction, z burnt during time t after ignition and the unburnt web fraction f at time t. It is derived from geometric consideration of the grain shape and can be expressed as

$$z = (1 - f)\ (1 + \theta f) \tag{1}$$

The Burning Law

The burning rate is represented by Vielle's law

$$D.\ \frac{df}{dt} = -\beta p_b^{\alpha} \tag{2}$$

The Energy Equation

The energy equation is given by

$$F.C.z = p_m.\left[k_0 + Ax - \frac{C}{\delta} - \left(V_c - \frac{1}{\delta}\right) C.z\right] + (\gamma - 1)W \tag{3}$$

W accounts for the work done by the reaction products in providing kinetic energy to the shot and propellant gases as well as the dissipation in overcoming bore resistance and heat transfer to the gun barrel.

Gas Motion behind the Moving Shot

We shall consider the motion of the gas, produced due to the burning of the propellant behind the moving shot. The equations of continuity is

$$\frac{\delta \rho}{\partial t} = \rho \frac{\partial u}{\partial y} = \frac{C.\frac{dz}{dt}}{A(x+l) + \frac{Cz}{\delta}} \tag{4}$$

Also equation of conservation of momentum is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} = -\frac{1}{\rho}.\frac{\partial p}{\delta y} \tag{5}$$

Where $u(y, t)$ is the gas velocity at a distance y at any time t after the movement of the shot, from the branch.

Here $\dfrac{C.\frac{dz}{dt}}{A(x+l) + \frac{Cz}{\delta}}$ is the mass per unit volume produced due to burning the charge fraction z.

This term acts as a source function.

To make the equations dimensionless the following transformations are used

$$\left.\begin{array}{lll} \xi = 1 + \dfrac{x}{l}, & Al = K_0 - \dfrac{C}{\delta}, & V_0 = \dfrac{\beta FC}{AD}, \\ \zeta = \dfrac{PAl}{FC}, & \zeta_b = \dfrac{p_b Al}{FC}, & \zeta_s = \dfrac{p_s Al}{FC}, \\ \zeta_m = \dfrac{p_m Al}{FC}, & \tau = \dfrac{V_0 t}{l}, & \eta = \dfrac{V}{V_0}, \end{array}\right\} \tag{6}$$

Put $Y = y/l$, $U = u/V_0$ and $\alpha = C/Al\delta$.

From solution of the equations (4) and (5) and using (6), we get

$$U = \frac{\eta[Y - Y_b] + [Y - Y_s]\,\alpha\,\frac{dz}{d\tau}}{(\xi + \alpha z)} \tag{7}$$

and

$$\zeta = \zeta_s - \frac{\epsilon z}{2M_1(\xi + \alpha z)}\left[2(Y - Y_s)\frac{d\eta}{d\tau} + \frac{(Y - Y_s)^2}{(\xi + \alpha z)}\left\{\frac{d\eta}{d\tau} + \alpha\frac{d^2 z}{d\tau^2}\right\}\right] \tag{8}$$

where

$$\epsilon = \frac{C}{1 \cdot 05\, m}, \qquad M_1 = \frac{FC}{1 \cdot 05\, m\, V_0^2}$$

At $\quad Y = Y_s \;,\; U = \eta$ and $\zeta = \zeta_s$

Also at $\quad Y = Y_b \;,\; \zeta = \zeta_b$

So equation (8) gives

$$\zeta_b = \zeta_s + \frac{\epsilon\, z}{2M_1} \left[\frac{d\eta}{d\tau} - \alpha \frac{d^2 z}{d\tau^2} \right] \tag{9}$$

Using value of ζ from equation (8) and integrating, we get

$$\zeta_m = \frac{1}{(Y_s - Y_b)} \int_{Y_b}^{Y_s} \zeta\, d\, Y = \zeta_s + \frac{\epsilon\, z}{6M_1} \left[2 \frac{d\eta}{d\tau} - \alpha \frac{d^2 z}{d\tau^2} \right] \tag{10}$$

Now kinetic energy of the propellant gas is

$$E_p = \frac{1}{2} \int_{y_b}^{y_s} \rho\, u^2 A\, dy$$

This gives

$$E_p = \frac{C V_0^2 z}{6} \left[\eta^2 + \alpha^2 \left(\frac{dz}{d\tau} \right)^2 - \alpha\, \eta \frac{dz}{d\tau} \right] \tag{11}$$

kinetic energy of the shot $= \frac{1}{2} mV^2$ and frictional losses due to bore resistance can be assumed equivalent to [0·05 × kinetic-energy of the shot].

Thus we have

$$W = \frac{1 \cdot 05}{2} mV^2 + \frac{C z V_0^2}{6} \left[\eta^2 + \alpha^2 \left(\frac{dz}{d\tau} \right)^2 - \alpha\, \eta \frac{dz}{d\tau} \right] + E_h$$

Energy losses due to heat transfer can be assumed as

$$E_h = K_H \frac{1 \cdot 05\, m V_0^2}{2} \eta^2$$

Let

$$(\bar{\gamma} - 1) = (\gamma - 1)(1 + K_H) \text{ and } \delta^1 = \frac{\epsilon\, (\bar{\gamma} - 1)}{6 M_1 (1 + K_H)}$$

Therefore

$$\frac{W}{FC} = \frac{1.05\, m V_0^2}{2\, FC} \left[\left(1 + K_H + \frac{\epsilon\, z}{3} \right) \eta^2 + \frac{\epsilon\, z\, \alpha}{3} \frac{dz}{d\tau} \left(\alpha \frac{dz}{d\tau} - \eta \right) \right]$$

So energy equation (3) becomes

$$z = \zeta_m (\xi - B\, Z) + \frac{(\gamma - 1)}{2\, M_1} \eta^2 + \delta^1 z \left[\eta^2 - \alpha\, z \frac{dz}{d\tau} + \alpha^2 \left(\frac{dz}{d\tau} \right)^2 \right] \tag{12}$$

Equation of Motion of the Shot

After accounting for the bore resistance the equation of motion of the shot in non-dimensional form can be written as

$$\frac{d\eta}{d\tau} = M_1 \zeta_s \tag{13}$$

Also we have

$$\frac{d\xi}{d\tau} = \eta \tag{14}$$

Final Equations of Internal Ballistics

By taking η as independent variable the equations of internal ballistics can be written as

$$\frac{dz}{d\eta} = \frac{\sqrt{(1+\theta)^2 - 4\theta z}}{M_1} (\zeta_b/\zeta_s) \tag{15}$$

$$\frac{d\xi}{d\eta} = \frac{\eta}{M_1 \zeta_s} \tag{16}$$

$$\zeta_b = \zeta_s \left(1 + \frac{\epsilon z}{2}\right) - \frac{\epsilon \alpha z M_1}{2} \zeta_s \frac{d}{d\eta}\left(\zeta_s \frac{dz}{d\eta}\right) \tag{17}$$

$$\zeta_m = \zeta_s \left(1 + \frac{\epsilon z}{3}\right) - \frac{\epsilon \alpha z M_1}{6} \zeta_s \frac{d}{d\eta}\left(\zeta_s \frac{dz}{d\eta}\right) \tag{18}$$

and

$$z = (\xi - B z)\zeta_m + \frac{(\bar{\gamma} - 1)}{2M_1} \eta^2 + \delta^1 z M_1 \zeta_s \frac{dz}{d\eta} \left\{ \frac{\eta^2}{M_1 \zeta_s \frac{dz}{d\eta}} + \alpha^2 M_1 \zeta_s \frac{dz}{d\eta} - \eta \alpha \right\} \tag{19}$$

Initial Conditions

At $\tau = 0$, $\eta = 0$, $\xi = 1$ and $\zeta_s = \zeta_{ss}$. In addition to this for the numerical solution of equations (15) to (19) we require the value of either ζ_b or z_i

Numerical Solution

Since either ζ_b or z_i are not easily determined experimently the system of equations (15) to (19) can be replaced by a new system

$$\frac{dz}{d\eta} = \frac{\sqrt{(1+\theta)^2 - 4\,\theta\, z}}{M_1} (\zeta_b/\zeta_s) \tag{20}$$

$$\frac{d\xi}{d\eta} = \frac{\eta}{M_1 \zeta_s} \tag{21}$$

$$\zeta_b = \zeta_s \left(1 + \frac{\epsilon z}{2}\right) \tag{22}$$

$$\zeta_m = \zeta_s \left(1 + \frac{\epsilon z}{3}\right) \tag{23}$$

$$z = (\xi - B z)\zeta_m + \frac{(\bar{\gamma} - 1)}{2M_1} \eta^2 \tag{24}$$

In this system z_i can be determined in terms of shot start pressure from (23) and (24). By Runge-Kutta method numerical solution of equations (20) to (24), with initial conditions $\xi = 1$, $\zeta_s = \zeta_{ss}$ and z_i determined above, is found. The computed values of $\zeta_b, z, \xi, \frac{dz}{d\eta}$ ζ_s, ζ_m are used to find the left out terms

$$\left.\begin{aligned} A' &= \frac{\epsilon \alpha z M_1}{2} \zeta_s \frac{d}{d\eta}\left(\zeta_s \frac{dz}{d\eta}\right) = \frac{\zeta_s}{2}\left[\epsilon z + 2 - \frac{2 M_1}{\sqrt{(1+\theta)^2 - 4\theta z}} \frac{dz}{d\eta}\right] \\ B' &= \frac{\epsilon \alpha z M_1}{6} \zeta_s \frac{d}{d\eta}\left(\zeta_s \frac{dz}{d\eta}\right) = \frac{A'}{3} \\ \text{and} \\ G &= \delta^1 z M_1 \zeta_s \frac{dz}{d\eta}\left\{\frac{\eta^2}{M_1 \zeta_s \frac{dz}{d\eta}} + \alpha^2 M_1 \frac{dz}{d\eta} - \eta \alpha\right\} \end{aligned}\right\} \quad (25)$$

in equations (17), (18), (19). The values for $\zeta_b, \zeta_m, \zeta_s$ are corrected accordingly.

$$(\zeta_b)_{corrected} = (\zeta_b)_{calculated} - A' \quad (26)$$

$$(\zeta_m)_{corrected} = (\zeta_m)_{calculated} - B' \quad (27)$$

$$(\zeta_s)_{corrected} = \frac{3\left[z - \frac{(\bar{\gamma}-1)}{2M_1}\eta^2 - G\right]}{(\xi - Bz)\cdot\left[\frac{M_1}{\sqrt{(1+\theta)^2 - 4\theta z}}\frac{dz}{d\eta} + \frac{(4+\epsilon z)}{2}\right]} \quad (28)$$

Also density of the reaction gas can be calculated from equation

$$\rho = \frac{z}{\left(\frac{Al}{C}\right)\cdot(\xi + \alpha z)} \quad (29)$$

After all burnt

By using computed values, we can get values of ξ, ζ_b, ζ_m, ζ_s, ρ and η when all the propellant is burnt, i.e. at $z = 1$.

Solution after all burnt is given by

$$\zeta_m = \frac{\zeta_{mb}\cdot(\xi_b - B)^{[1+(\bar{\gamma}-1)/(1+\epsilon/3)]}}{(\xi - B)^{[1+(\bar{\gamma}-1)/(1+\epsilon/3)]}} \quad (30)$$

and

$$\eta^2 = \left[1 - \zeta_m\cdot(\xi - B)\right]\frac{2M_1}{(\bar{\gamma}-1)} \quad (31)$$

where ζ_{mb} and ξ_b are the values of ζ_m and ξ at all burnt.

Equations (30), (31) gives ζ_m and η for any value of ξ after all burnt. Using these, we have

$$\zeta_s = \frac{\zeta_m}{(1+\epsilon/3)} \quad (32)$$

and

$$\zeta_b = \zeta_m\cdot\frac{(1+\epsilon/2)}{(1+\epsilon/3)} \quad (33)$$

TABLE 1

EXPERIMENTAL DATA

Particulars	Gun No. 1	Gun No. 2
Barrel length	586·232 cm	395·986 cm
Vol. of the chamber	18697·567 cu cm	3519·9276 cu cm
Bore area	135·4836 sq cm	46·516036 sq cm
Force constant	10059·2128395 cm-tonne/Kg	9558·5280375 cm-tonne/Kg
Charge wt.	13·7996567 Kg	2·6798268 Kg
Co-volume	1004·3336216 cu cm/Kg	1003·6110794 cu cm/Kg
Reciprocal of propellant density	603·3227151 cu cm/Kg	609·8255947 cu cm/Kg
Shape factor	—0·15	—0·172
Web size	0·2159 cm	0·110998 cm
burning rate	0·2479887 cm/sec/tonne sq cm	0·2270646 cm/sec/tonne sq cm
γ	1·27	1·26
K_H	0·1	0·1
Shell wt.	33·5159793 Kg	5·9701897 Kg
Short start pressure	19·6654077 tonne sq cm	29·4981115 tonne sq cm

TABLE 2

RESULTS OBTAINED FROM HUNT-HIND'S METHODS AND THIS METHOD

	Particulars	Hunt-Hind's method	This method
Gun No. 1	Muzzle Velocity	97707·8636753 cm/sec	98180·5995746 cm/sec
	Maximum Pressure	168·0411493 tonne sq cm	163·0921240 tonne sq cm
	All burnt position	190·1573506 cm	207·0010277 cm
Gun No. 2	Muzzle velocity	104735·2091588 cm/sec	105781·8683352 cm/sec
	Maximum Pressure	173·5292782 tonne sq cm	168·0975635 tonne sq cm
	All burnt position	103·4649160 cm	112·9863394 cm

NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

Results obtained by applying the above numerical technique and the Hunt-Hind's method, using two gun data given in Table 1, are presented in Table 2. Using these two techniques, pressure distribution for two guns were found. Fig. 1 and Fig. 5 shows the distribution of mean pressure against the distance travelled by the shot, for each gun. It is observed that maximum pressure obtained for each gun, using the

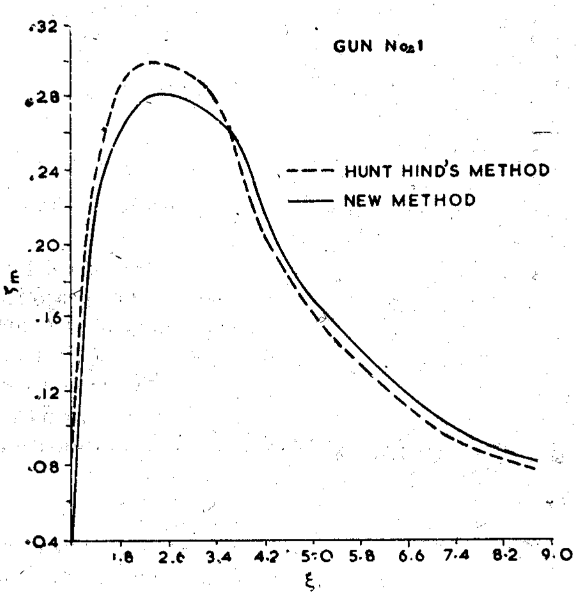

Fig. 1—Variation of the mean pressure with the distance travelled by the shot.

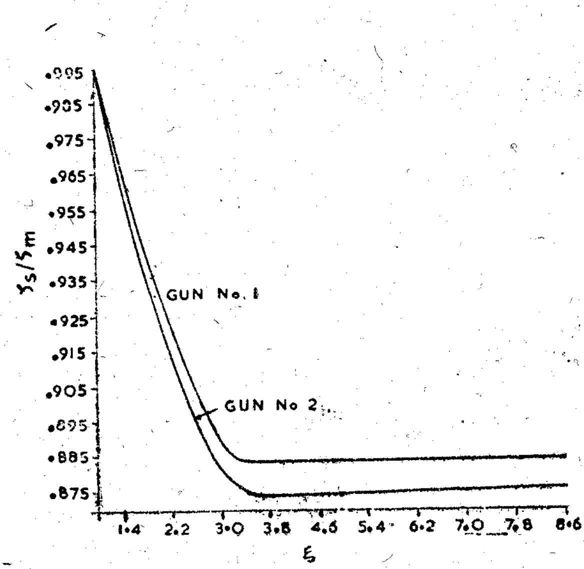

Fig. 2—Variation of the ratio of the shot base pressure to the mean pressure with the distance travelled by the shot.

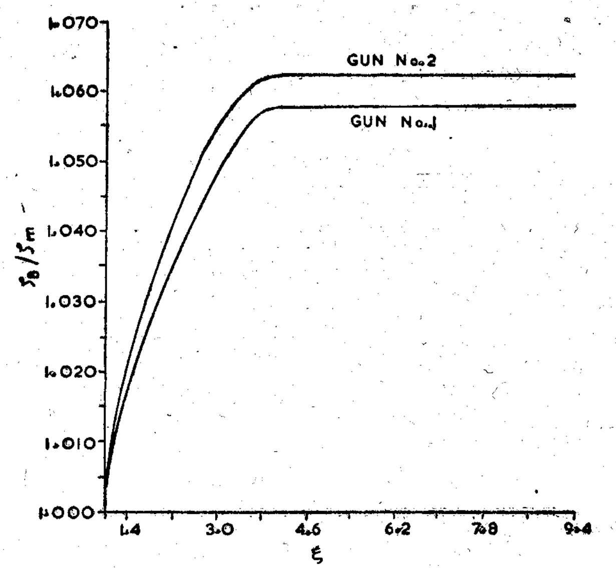

Fig. 3—Variation of the ratio of the breech pressure to the mean pressure with the distance travelled by the shot.

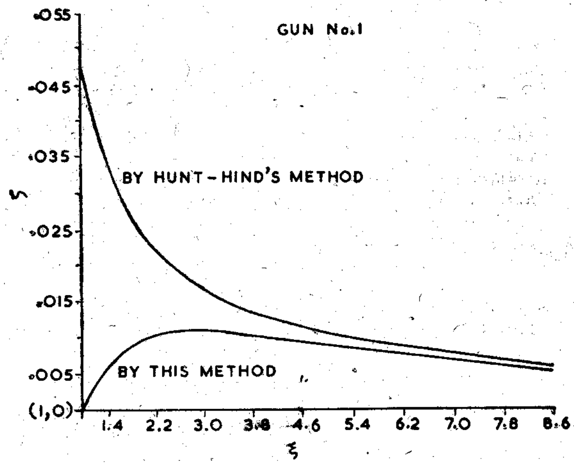

Fig. 4—Variation of the density of the propellant gas with the distance travelled by the shot.

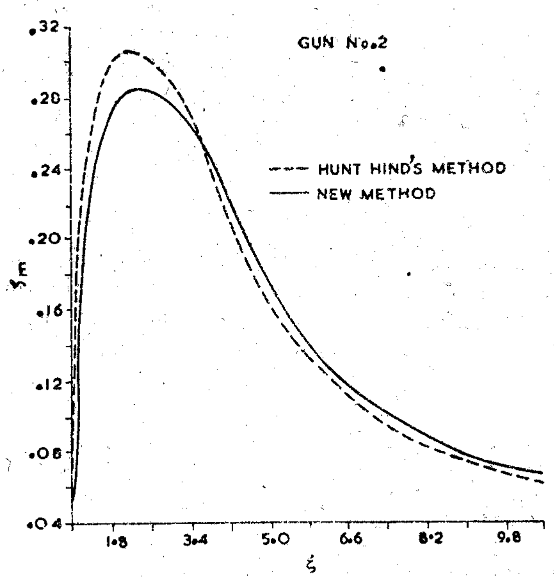

Fig. 5—Variation of the mean pressure with the distance travelled by the shot.

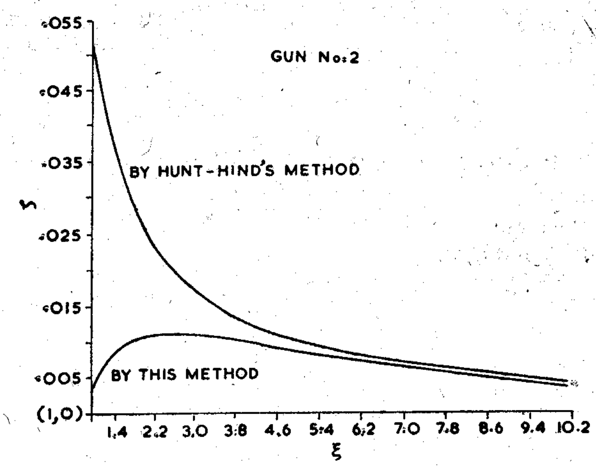

Fig. 6—Variation of the density of the propellant gas with the distance travelled by the shot.

new method, is less than the Hunt-Hind's predicted maximum pressure. Also initially mean pressure is high in the Hunt-Hind's method than this method but after all burnt it is low. Fig. 2 presents the graphical representation of the ratio of the shot base pressure to the mean pressure against displacement of the shot, for each gun. Also in Fig. 3 the breech pressure to the mean pressure ratio has been plotted against the displacement of the shot. Fig 4 and Fig. 6 presents density curves for both the guns. Density curve based on the Hunt-Hind's method shows that, the density of the reaction products is maximum initially and then decreases continuously. On the other hand density curve based on method presented shows that density of the propellant gas increases in the initial stage of the movement of the shot and then slowly decreases.

CONCLUSION

The maximum breech pressures predicted by the new system are less than that of Hunt-Hind's method. All burnt position, predicted by the Hunt-Hind's method, occurs early than the predicted all burnt position of this new method. Muzzle velocities calculated by the new system are very much close to the Hunt-Hind's predicted velocities.

Hence, it is concluded that the technique presented is capable of accurately simulating gun-cycle (distance travelled, pressure, energy, density................as a function of velocity) for any loading conditions.

ACKNOWLEDGEMENT

The author is thankful to Dr. J. N. Nanda, Director and Dean, Institute of Armament Technology, Pune for his permission to publish the paper. The author also wishes to express his gratitude to Dr. C. D. Ghildyal, Principal Scientific Officer, Institute of Armament Technology, Pune for his valuable guidance in the preparation of this paper.

REFERENCES

1. 'Internal Ballistics' (Her Majesty's Stationary Off), 1951.
2. CLEMMOW, C. A., *Phil. Trans. Roy Soc.*, **A 227** (1928), 345.
3. CHUGH, O. P., *Mem. Artillery*, **41** (1967), 247.
4. CHUGH, O. P., *Explosivestoffe*, **15** (1967), 250.
5. CHUGH, O. P., *Mem. Artillery*, **43** (1969), 195.
6. PRASAD, G. R., *Mem. Artillery*, **44** (1970), 195.
7. PRASAD, G. R., *Mem. Artillery*, **43** (1969), 793.
8. AGGARWAL, S. P., MODI, K. K., & VARMA, P. S., *Mem. Artillery*, **44** (1970), 415.
9. AGGARWAL, S. P., VARMA, P. S., & MODI, J. K., *Mem. Artillery*, **44**, (1970), 839.