ON THE FLOW OF A DUSTY FLUID

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The present paper deals with the flow of a fluid with uniform distribution of dust particles in two parts. In part A, the motion induced in the dusty fluid is considered in the case when the plane moves parallel to itself impulsively from rest with uniform velocity; for this case, the velocity profiles of fluid as well as dust particles are obtained. In part B, the flow is produced by the motion of cylinders. Two cases are taken : (*i*) cylinders moving exponentially with time and, (*ii*) cylinders moving in simple harmonic motion.

The study of the fluid flow containing solid particles has been the subject of scientific and engineering research for a long time. The interest in problems of flow of a dusty fluid (*i.e.* a mixed system of fluid and dust particles) has increased in recent years. Situations which occur frequently are concerned with the motion of a liquid or gas which contains a distribution of solid particles. The mathematical description of such diverse systems must of course vary widely. In order to formulate the problem in a reasonably simple manner and to bring out the essential features, we make simplifying assumptions about the motion of fluid and dust particles. The fluid is incompressible and the particles are spheres and uniform in size. The number of the particles is so large that system of the particles can be considered as continuous medium. The particle—particles interaction and the bulk concentration of the particles are negligible. The fluid particle interaction is according to Stokes drag law. We denote E to be the radius of the particles, ν the kinematic viscosity of the fluid, ρ the density of the fluid and m the mass of each particle. In the case of spherical particles K, the constant of proportionality in Stokes law is equal to $6\pi\rho\nu E$. In term of these quantities the relaxation time τ, which is a measure of the time taken by the dust particles to adjust to changes in the fluid velocity is given $\tau = m/K$.

FORMULATION

The equations governing the unsteady motion of an incompressible viscous fluid with uniform distribution of dust particles are[1]

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{1}{\rho}\,\overrightarrow{\text{grad}}\,p + \nu\nabla^2\vec{u} + \frac{KN}{\rho}\,(\vec{v}-\vec{u}) \tag{1}$$

$$\text{div}\,\vec{u} = 0 \tag{2}$$

$$\tau\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = \vec{u} - \vec{v} \tag{3}$$

$$\frac{\partial N}{\partial t} + \text{div}\,N\vec{v} = 0, \tag{4}$$

where $\vec{u}$ and $\vec{v}$ are the velocities of fluid and dust particles respectively, N is the number density of dust particle each of mass m; K is the Stokes resistance coefficient.

As both m and ρ are uniform N can be taken as constant and equal to N_0. It can be seen from (3) that as $\tau \to 0$ the particles follow the fluid motion exactly.

Part A

We assume that fluid containing a uniform distribution of dust particles occupies the semi-infinite space above a rigid plane boundary. The motion induced in the dusty fluid is considered in the case when the plane moves parallel to itself impulsively from rest with uniform velocity. Our aim is to derive the expressions for the velocity of fluid as well as dust particles. We also assume that the motion is induced by a prescribed velocity of the rigid plane boundary at $y=0$ parallel to itself in the x-direction.

For the present case, we have

$$u_1 = u_1(y,t), \qquad u_2 = 0, \qquad u_3 = 0,$$
$$v_1 = v_1(y,t), \qquad v_2 = 0, \qquad v_3 = 0,$$

where (u_1, u_2, u_3) and (v_1, v_2, v_3) are the components of the fluid and dust particles respectively. The equations of motion then become

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} + \frac{h}{\tau}(v_1 - u_1) \tag{5}$$

$$\tau \frac{\partial v_1}{\partial t} = u_1 - v_1, \tag{6}$$

where $h = mN_0/\rho$ is the mass concentration of the dust particles.

Using the dimensionless time variable t/τ and a dimensionless length $y/\sqrt{\nu\tau}$, the above equations become

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial y^2} + h(v_1 - u_1) \tag{7}$$

$$\frac{\partial v_1}{\partial t} = u_1 - v_1, \tag{8}$$

where t and y are now dimensionless.

The boundary conditions are

$$\left.\begin{aligned} t = 0,\ u_1 = 0,\ \ v_1 = 0 \text{ for } y > 0 \\ t > 0,\ u_1 = 1 \text{ at } y = 0 \\ u_1 \to 0 \text{ as } y \to \infty \end{aligned}\right\} \tag{9}$$

We now define the Laplace transform ot u_1 and v_1 by

$$\left.\begin{aligned} \bar{u}_1 = \int_0^\infty u_1\, e^{-st}\, dt \\ \qquad\qquad \mathrm{Re}\,(s) > 0 \\ \bar{v}_1 = \int_0^\infty v_1\, e^{-st}\, dt \end{aligned}\right\} \tag{10}$$

In terms of (10) from (7) and (8), we have

$$\frac{d^2 \bar{u}_1}{dy^2} - \frac{s(1+h+s)}{1+s}\bar{u}_1 = 0 \tag{11}$$

$$\bar{v}_1 = \frac{\bar{u}_1}{1+s}. \tag{12}$$

The boundary conditions on transformation become

$$\begin{aligned} u_1 &= 0 \text{ as } y \to \infty \\ u_1 &= 1/s \text{ at } y = 0 \text{ for } t > 0. \end{aligned} \tag{13}$$

Solving for u_1 and applying boundary conditions (13), we find

$$\bar{u}_1 = \frac{1}{s}\exp\left[-y\sqrt{\frac{s(1+h+s)}{1+s}}\right]. \tag{14}$$

Substituting this value of u_1 in equation (12), we get

$$\bar{v}_1 = \frac{1}{s(s+1)}\exp\left[-y\sqrt{\frac{s(1+h+s)}{1+s}}\right]. \tag{15}$$

For small values of t which correspond to large values of s, on inversion we obtain

$$u_1 = \frac{1}{2}\left[e^{-y\sqrt{h}} \ \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{ht}\right) + e^{y\sqrt{h}} \ \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{ht}\right)\right] \tag{16}$$

$$v_1 = u_1 - \frac{1}{2} e^{-t}\left[{}^{-y\sqrt{h-1}} \ \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{(h-1)t}\right) + \right.$$

$$\left. + e^{y\sqrt{h-1}} \, \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{(h-1)t}\right)\right]. \tag{17}$$

For large values of t i.e. $s << 1$, we obtain

$$u_1 = \mathrm{erfc}\frac{y\sqrt{h-1}}{2\sqrt{t}} \tag{18}$$

$$v_1 = u_1 - \frac{1}{2} \ e^{-t}\left[e^{-i\sqrt{h-1}} \ \mathrm{erfc}\left(\frac{\sqrt{h-1}}{2\sqrt{t}} - i\,t\right) + \right.$$

$$\left. + e^{i\sqrt{h-1}} \, \mathrm{erfc}\left(\frac{\sqrt{h-1}}{2\sqrt{t}} + i\,t\right)\right]. \tag{19}$$

From the expressions (16) to (19) it is obvious that velocity of the fluid particles is greater than that of the dust particles.

Part B

Let the radii of the coaxial circular cylinders be R_1, R_2 $(R_2 > R_1)$. We take cylindrical polar system of coordinates (r, θ, z) with z-axis along the axis of the cylinders.

Then for the present case, we have

$$\left.\begin{array}{l} u_r = 0, \quad u_\theta = 0, \quad u_z = u_z(r, t) \\ v_r = 0, \quad v_\theta = 0, \quad v_z = v_z(r, t), \end{array}\right\} \tag{20}$$

where (u_r, u_θ, u_z) and (v_r, v_θ, v_z) are the components of the fluid and dust particles respectively. The fluid motion is due entirely to unsteady motion of the cylinders, the pressure for upstream and for downstream being kept equal throughout the motion. Thus $(1/\rho)\ \partial p/\partial z = 0$ for such case, and equations of motion then become

$$\frac{\partial u_z}{\partial t} = \nu\left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\ \frac{\partial u_z}{\partial r}\right] + \frac{h}{\tau}\ (v_z - u_z) \tag{21}$$

$$\tau \frac{\partial v_z}{\partial t} = u_z - v_z \tag{22}$$

Let

$$u_z = u(r) \ \exp\left\{(i\sigma - \lambda^2)\ t\right\} \tag{23}$$

$$v_z = v(r) \ \exp\left\{(i\sigma - \lambda^2)\, t\right\}, \tag{24}$$

where σ and λ are real constants and $0 \leqslant \lambda < 1$.

Let the boundary conditions be

$$u_z = w_1 \ \exp\left\{(i\sigma - \lambda^2)\ t\right\}, \quad r = R_1 \tag{25}$$

$$v_z = w_2 \ \exp\left\{(i\sigma - \lambda^2)\ t\right\}, \quad r = R_2 \tag{26}$$

Substituting (23) and (24) in equations (21) and (22), we have

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \beta^2 u = 0 \tag{27}$$

$$v = u/[1 + (i\sigma - \lambda^2)\tau], \tag{28}$$

where

$$\beta^2 = -\frac{(i\sigma - \lambda^2)}{\nu}\left[\frac{1 + h + (i\sigma - \lambda^2)\tau}{1 + (i\sigma - \lambda^2)\tau}\right]. \tag{29}$$

Solution of (27) is

$$u = A\, J_0(\beta r) + B\, Y_0(\beta r). \tag{30}$$

From (30) and (23), we have

$$u_z = \left[A\, J_0(\beta r) + B\, Y_0(\beta r)\right] \exp\left\{(i\sigma - \lambda^2)\, t\right\}. \tag{31}$$

Eliminating A and B with the help of boundary conditions (25) and (26), we get

$$u_z = \left[\frac{w_1 C_0(r, R_2, \beta) + w_2 C_0(R_1, r, \beta)}{C_0(R_1, R_2, \beta)}\right] \exp\left\{(i\sigma - \lambda^2)\, t\right\}, \tag{32}$$

where

$$C_0(R_1, R_2, \beta) = J_0(\beta R_1)\, Y_0(\beta R_2) - J_0(\beta R_2)\, Y_0(\beta R_1).$$

If $\sigma = 0$, we have

$$u_z = \left[\frac{w_1 C_0(r, R_2, \beta_1) + w_2 C_0(R_1, r, \beta_1)}{C_0(R_1, R_2, \beta_1)}\right] e^{-\lambda^2 t} \tag{33}$$

and

$$v_z = \left[\frac{w_1 C_0(r, R_2, \beta_1) + w_2 C_0(R_1, r, \beta_1)}{(1 - \lambda^2\tau)\, C_0(R_1, R_2, \beta_1)}\right] e^{-\lambda^2 t} \tag{34}$$

where

$$\beta_1^2 = \frac{\lambda^2}{\tau}\left(\frac{1 - \lambda^2\tau + h}{1 - \lambda^2\tau}\right).$$

These are the expressions for the velocity of fluid and dust particles when cylinders move exponentially with time.

If $\lambda = 0$, we get

$$u_z = \left[\frac{w_1 C_0(r, R_2, \beta_2) + w_2 C_0(R_1, r, \beta_2)}{C_0(R_1, R_2, \beta_2)}\right] e^{i\sigma t} \tag{35}$$

$$v_z = \left[\frac{w_1 C_0(r, R_2, \beta_2) + w_2 C_0(R_1, r, \beta\ )}{(1 + i\sigma\tau)\, C_0(R_1, R_2, \beta_2)}\right] e^{i\sigma t} \tag{36}$$

where

$$\beta_2^2 = -\frac{1}{\nu}\left[\frac{h\sigma\tau + i\sigma(1 + h + \sigma^2\tau^2)}{1 + \sigma^2\tau^2}\right].$$

These are the expressions for the velocity of fluid and dust particles when the cylinders are in simple harmonic motion.

From (33) and (34), it is clear that the velocity of the dust particles is greater than that of the fluid when the cylinders move exponentially with time. In the case when the cylinder execute simple harmonic motion the dust velocity is less than that of the fluid. When the dust is very fine, the relaxation time of the dust decreases and ultimately as $\tau \to 0$ the dust particles follow the fluid motion exactly i.e. if the masses of the dust particles are small enough their influence on the fluid is reduced and in the limit as $m \to 0$ the fluid becomes ordinary viscous and we get the expressions for the velocity of viscous fluid in the two above mentioned cases.

REFERENCE

1. SAFFMAN, P. G., *J. Fluid Mech.*, **13** (1962), 120.