OSCILLATORY FREE CONVECTION FROM A ROTATING DISK

R.C. BHATTACHARJEE

Dibrugarh University, Dibrugarh

(*Received* 5 *February*, 1976)

Free convection flow from an infinite disk, when both the disk and the surrounding fluid rotate with the same angular velocity about the same vertical axis has been considered. The disk is maintained at an axisymmetric nonuniform temperature distribution which oscillates about a steady nonzero mean. The temperature of the fluid at large distance from the disk is taken to be constant. The problem is formulated for small values of the reduced frequency which is the ratio of the frequency of oscillations of the temperature of the disk to the angular velocity of the fluid. The equations are integrated by Ka'rma'n-Pohlhausen method for a fluid of Prandtl number unity. The solution holds good when the Rossby number and the reduced frequency are of the same order.

It has been seen by Srivastava & Bhattacharjee[1] that if a disk rotates with a constant angular velocity Ω about an axis perpendicular to its plane in a fluid which is rotating at infinity with the same angular velocity about the same axis and if the disk is maintained at an axisymmetric nonuniform temperature distribution with a minimum at the point of intersection of the disk with the axis (pole), then a special type of free convection flow occurs. In this paper we have extended the above mentioned problem, when the axis-symmetric nonuniform temperature distribution of the disk performs oscillations with frequency ω about a steady nonzero mean. Using Lighthill's[2] technique we have written the velocity field $\vec{V}$ as $\vec{V} = \vec{V}_s + \epsilon \vec{V}_f e^{i\omega t}$ and the temperature distribution as $T = T_s + \epsilon T_f e^{i\omega t}$ and retained only the first power of ϵ for the fluctuating part of the velocity and the temperature field, where $\vec{V}_s$, $\vec{V}_f$, T_s and T_f are independent of time.

The steady flow has been discussed in detail in[1]. Using these solutions the fluctuating part of the and the temperature field has been obtained for small values of the reduced frequency $\lambda (= \omega/\Omega)$. equations of motion have been integrated by Ka'rma'n-Pohlhausen method for a fluid of Prandtl number unity and for small Rossby number. It is seen that the solution holds good when λ and the Rossby β, are of the same order.

EQUATIONS OF MOTION

Consider a disk, $z = 0$, rotating with a constant angular velocity Ω about an axis perpendicular to its plane and a viscous conducting fluid filling the space $z > 0$. The fluid at infinity is also rotating with a constant angular velocity Ω in the same sense as the disk and about the same vertical axis. Let a cylindrical polar coordinate system (r, θ, z) be fixed in the rotating disk with the origin at the point of intersection of the disk and the axis of rotation, and with the z axis pointing vertically upwards. The disk is maintained at a temperature

$$T_w = T_0 + (T^* - T_0)(1 + \epsilon \cos \omega t) \frac{r^2 \Omega}{\nu} \tag{1}$$

and the fluid at $z = \infty$ is maintained at a constant temperature T_∞ where T_0 is the constant temperature at the pole, T^* is the mean temperature at $r = (\nu/\Omega)^{\frac{1}{2}}$, ν is the kinematic coefficient of viscosity and t is time. The point of intersection of the disk and the axis of rotation is named as pole. $(T^* - T_0)$ is very small so that T_w does not exceed T_∞ in the region under consideration.

Taking Boussinesq approximation for density variation and writing

$$p = \frac{P}{\rho_0} - g z - \frac{1}{2} \Omega^2 d^2$$

where P is the pressure in the fluid, d is the distance of a particle of fluid from the axis of rotation and proceeding in the same way as in[1], the equations of motion can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - 2v\Omega = -\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r}\right] \quad (2)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + 2u\,\Omega = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r}\right] \quad (3)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] + g\,\alpha\,(T - T_0) \quad (4)$$

where u, v, w are velocity components in the directions of r, θ, z respectively.

The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \quad (5)$$

Neglecting the viscous dissipation the equation for the temperature is given by

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = K\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \quad (6)$$

where K is the thermal diffusivity.

The boundary conditions on the velocity components and the temperature are given by

$$u = v = w = 0 \ \text{ at } z = 0\,; \quad u \to 0\,, \quad v \to 0 \text{ as } z \to \infty\,. \quad (7)$$

and

$$\left.\begin{aligned} T &= T_0 + (T^* - T_0)\,(1 + \epsilon \cos \omega\, t)\,\frac{r^2 \Omega}{\nu} \ \text{ at } z = 0 \\ T &\to T_\infty \text{ as } z \to \infty \end{aligned}\right\} \quad (8)$$

Similarity Transformation

In view of the boundary conditions (7) and (8), we assume the following similarity transformations

$$u = r\,\Omega\,\frac{\partial F(\xi,\tau)}{\partial \xi}\,,\ v = r\,\Omega\,G(\xi,\tau)\,,\ w = -2\sqrt{\nu\,\Omega}\,F(\xi,\tau)$$

$$\frac{T - T_0}{T^* - T_0} = \chi(\xi,\tau) + \frac{r^2\,\Omega}{\nu}\,\phi(\xi,\tau)\,, \quad (9)$$

where $\xi = z\,(\Omega/\nu)^{\frac{1}{2}}$ and $\tau = \omega\, t$. Following Lighthill's[2] method of analysis, we write

$$\left.\begin{aligned} F(\xi,\tau) &= F_0(\xi) + \epsilon\, F_1(\xi)\, e^{i\tau} \\ G(\xi,\tau) &= G_0(\xi) + \epsilon\, G_1(\xi)\, e^{i\tau} \\ \chi(\xi,\tau) &= \chi_0(\xi) + \epsilon\, \chi_1(\xi)\, e^{i\tau} \\ \phi(\xi,\tau) &= \phi_0(\xi) + \epsilon\, \phi_1(\xi)\, e^{i\tau} \end{aligned}\right\} \quad (10)$$

Thus u, v, w and T are taken as

$$u = r\,\Omega\,[F_0'(\xi) + \epsilon\,F_1'(\xi)\,e^{i\tau}] \tag{11}$$

$$v = r\,\Omega\,[G_0(\xi) + \epsilon\,G_1(\xi)\,e^{i\tau}] \tag{12}$$

$$w = -2\sqrt{\nu\Omega}\,[F_0(\xi) + \epsilon\,F_1(\xi)\,e^{i\tau}] \tag{13}$$

$$\frac{T - T_0}{T^* - T_0} = [\chi_0(\xi) + \epsilon\,\chi_1(\xi)\,e^{i\tau}] + \frac{r^2\,\Omega}{\nu}\,[\phi(\xi) + \epsilon\,\phi_1(\xi)\,e^{i\tau}] \tag{14}$$

where a prime denotes a differentiation with respect to ξ. We assume the following form of the pressure p

$$p = (\nu\,\Omega)\left[\left\{p_0(\xi) + \epsilon\,p_1(\xi)\,e^{i\tau}\right\} + \frac{r^2\,\Omega}{\nu}\left\{P_0(\xi) + \epsilon\,P_1(\xi)\,e^{i\tau}\right\}\right] \tag{15}$$

The functions F_0, G_0, p_0, P_0, χ_0, and ϕ_0 are real but the functions F_1, G_1, p_1, P_1, χ_1 and ϕ_1 are complex.

The boundary conditions in terms of the new variables are

$$\left.\begin{aligned} &F_0 = F_0' = G_0 = F_1 = F_1' = \chi_0 = \chi_1 = 0\,, \quad \phi_0 = \phi_1 = 1 \text{ at } \xi = 0\,. \\ &F_0' = G_0 = F_1' = G_1 = \chi_1 = \phi_0 = \phi_1 = 0\,, \quad \chi_0 = \frac{T_\infty - T_0}{T^* - T_0} = S \text{ as } \xi \to \infty \end{aligned}\right\} \tag{16}$$

Substituting these expressions for u, v, w, p and T in the equations (2) to (4) and in (6) and eliminating pressure terms from (2) and (4) we get the following equations (neglecting squares and higher powers of ϵ).

$$F_0'''' + 2\,(F_0\,F_0''' + G_0\,G_0' + G_0') = 2\,\beta\,\phi_0 \tag{17}$$

$$2\,(G_0\,F_0' - F_0\,G_0' + F_0') = G_0'' \tag{18}$$

$$2\,\sigma\,(\phi_0\,F_0' - F_0\,\phi_0') = \phi_0'' \tag{19}$$

$$\chi_0'' + 4\,(\phi_0 + 2\,\sigma\,F_0\,\chi_0') = 0 \tag{20}$$

$$4\,F_0\,F_0' = -\,p_0' - 2\,F_0'' + \beta\,\chi_0 \tag{21}$$

$$P_0' = \beta\,\phi_0\,, \tag{22}$$

$$-\lambda\,i\,F_1'' + F_1'''' + 2\,[F_0\,F_1''' + F_1\,F_0''' + G_0\,G_1' + G_1\,G_0' + G_1'] = 2\beta\,\phi_1 \tag{23}$$

$$\lambda\,i\,G_1 + 2\,(G_0\,F_1' + G_1\,F_0' - F_0\,G_1' - F_1\,G_0' + F_1') = G_1'' \tag{24}$$

$$\lambda\,i\,\sigma\,\chi_1 - 2\sigma\,(F_0\,\chi_1' + F_1\,\chi_0') = 4\,\phi_1 + \chi_1'' \tag{25}$$

$$\lambda\,i\,\sigma\,\phi_1 + 2\sigma\,[\,(\phi_0\,F_1' + \phi_1\,F_0') - (F_0\,\phi_1' + F_1\,\phi_0')\,] = \phi_1'' \tag{26}$$

$$-\,2\,i\,\lambda\,F_1 + 4\,(F_0\,F_1' + F_1\,F_0') = -\,p_1' - 2\,F_1'' + \beta\chi_1 \tag{27}$$

$$P_1' = \beta\phi_1 \tag{28}$$

where $\sigma = \nu/K$ is the Prandtl number, $\beta = g\alpha\,(T^* - T_0)/\,\Omega\,\sqrt{\nu\Omega}$ is the Rossby number which is small and $\lambda = \omega/\Omega$ is the reduced frequency.

SOLUTION OF THE EQUATIONS

The equations (17) to (22) give the steady flow created by the non-uniform axis-symmetric and steady temperature distribution of the disk. These equations have been integrated by Kràmàn-Pohlhausen method for a fluid of Prandtl number unity and discussed in detail by Srivastava and Bhattacharjee[1]. The interaction of the steady and the fluctuating flows is governed by the equations (23) to (28). Using the solutions of (17) to (22) (see reference 1) we have determined the fluctuating part of the flow for small values of the reduced frequency λ. We expand the functions F_1, G_1, χ_1 and ϕ_1 in powers of $(i\lambda)$.

$$\begin{aligned} F_1(\xi) &= f_0(\xi) + i\lambda f_1(\xi) + \\ G_1(\xi) &= g_0(\xi) + i\lambda g_1(\xi) + \\ \phi_1(\xi) &= h_0(\xi) + i\lambda h_1(\xi) + \\ \chi_1(\xi) &= q_0(\xi) + i\lambda q_1(\xi) + \end{aligned} \tag{29}$$

Substituting the series (29) in the equations (23) to (28) and equating real and imaginary parts on both sides of the equation, we get the following sets of differential equations

$$f_0'''' + 2(F_0 f_0''' + f_0 F_0''' + G_0 g_0' + g_0 G_0' + g_0') = 2\beta h_0 \tag{30}$$

$$2(G_0 f_0' + g_0 F_0' - F_0 g_0' - f_0 G_0' + f_0') = g_0'' \tag{31}$$

$$2\sigma(F_0 q_0' + f_0 \chi_0') + 4h_0 + q_0'' = 0 \tag{32}$$

$$2\sigma(\phi_0 f_0' + h_0 F_0' - F_0 h_0' - f_0 \phi_0') = h_0'' \tag{33}$$

and

$$f_1'''' - f_0'' + 2(F_0 f_1''' + f_1 F_0''' + G_0 g_1' + g_1 G_0' + g_1') = 2\beta h_1 \tag{34}$$

$$g_0 + 2(G_0 f_1' + g_1 F_0' - F_0 g_1' - f_1 G_0' + f_1') = g_1'' \tag{35}$$

$$\sigma[q_0 - 2(F_0 q_1' + f_1 \chi_0')] = 4h_1 + q_1'' \tag{36}$$

$$\sigma[h_0 + 2(\phi_0 f_1' + h_1 F_0' - F_0 h_1' - f_0 \phi_0')] = h_1'' \tag{37}$$

Where we have neglected squares and higher powers of λ. The boundary conditions (16) can now be written as

$$\left.\begin{aligned} &f_0 = f_0' = f_1 = f_1' = g_0 = g_1 = q_0 = q_1 = h_1 = 0,\ h_0 = 1 \text{ at } \xi = 0 \\ &f_0' = f_1' = g_0 = g_1 = q_0 = q_1 = h_0 = h_1 = 0 \text{ as } \xi \to \infty \end{aligned}\right\} \tag{38}$$

The equations (30) to (33) give the quasi-steady solution f_0, g_0, q_0, h_0, while equations (34) to (37) give the first approximation in λ to the fluctuating components f_1, g_1, q_1 and h_1. These equations can be integrated by any numerical method. We have integrated them by Kàrmàn-Pohlhausen method for a fluid of Prandtl number unity in which case the thickness of the velocity and temperature boundary layers become equal, say $\delta = D(\nu/\Omega)^{\frac{1}{2}}$. Let η stands for the variable $\eta = \xi/D$ and suppose $M(\xi) = M(\eta D) = \bar{M}(\eta)$. We assume the following polynomials for f_j', g_j, q_j and h_j, $j = 0, 1$.

$$D f_0'(\xi) = \frac{d\bar{f}_0}{d\eta} = (A_0\eta + B_0\eta^2)(1-\eta)^4 \tag{39}$$

$$g_0(\xi) = \bar{g}_0(\eta) = C_0(\eta + 4\eta^2)(1-\eta)^4 \tag{40}$$

$$h_0(\xi) = \bar{h}_0(\eta) = \{1 + E_0\eta + (4E_0 - 6)\eta^2\}(1-\eta)^4 \tag{41}$$

$$q_0(\xi) = \bar{q}_0(\eta) = [(A_2 + SB_2)\eta + \{4(A_2 + SB_2) - 2D^2\}\eta^2](1-\eta)^4 \tag{42}$$

$$D f_1'(\xi) = \frac{d\bar{f}_1(\eta)}{d\eta} = (A_1\eta + B_1\eta^2)(1-\eta)^4 \tag{43}$$

$$g_1(\xi) = \bar{g}_1(\eta) = C_1(\eta \times 4\eta^2)(1-\eta)^4 \tag{44}$$

$$h_1(\xi) = \bar{h}_1(\eta) = \{E_1\eta + (4E_1 + \tfrac{1}{2}D^2)\eta^2\}(1-\eta)^4 \tag{45}$$

$$q_1(\xi) = \bar{q}_1(\eta) = (A_3 + SB_3)(\eta + 4\eta^2)(1-\eta)^4 \tag{46}$$

The values of the constants A_0, B_0, C_0, E_0, A_2, B_2, A_1, B_1, C_1, E_1, A_3, and B_3 are given in Table 1 for $\beta = 0.1, 0.2, 0.3$.

TABLE 1

VALUES OF DIFFERENT CONSTANTS FOR VARIOUS VALUES OF β

	β=0·1	β=0·2	β=0·3
D	8·6089	7·1126	6·1721
$-A_0$	9·7508	10·3121	9·8501
$-B_0$	33·5747	33·0993	28·0560
$-C_0$	0·0457	0·1972	0·3344
$-E_0$	0·1501	0·0336	0·1692
A_2	32·7783	23·4913	18·0880
B_2	0·6832	0·6420	0·6055
A_1	13·8524	8·0302	5·2147
$-B_1$	41·4550	14·7485	8·0117
$-C_1$	0·6041	0·1686	0·0102
$-E_1$	2·4720	2·0906	1·1611
$-A_3$	20·8476	18·4335	6·7850
$-B_3$	1·8157	0·8975	0·6812

DISCUSSION

The variations of f_0' and f_1' against $\xi = z\,(\Omega/\nu)^{\frac{1}{2}}$ have been shown in Fig. 1 for $\beta = 0.2, 0.3$. The variations of $-g_0$ and $-g_1$ against ξ have been shown in Fig. 2 for $\beta = 0.2, 0.3$. The solid lines indicate variations of f_0' and $-g_0$ whereas the broken lines indicate variations of f_1' and $-g_1$. The order of magnitudes indicate that $\lambda\ (= \omega/\Omega)$ should be sufficiently small and should be of the order of β, so that the series (29) converge. The heat transfer per unit area from the fluid to the plate is given by

$$q = k\left[\frac{\partial T}{\partial z}\right]_{z=0} = k\,(\Omega/\nu)^{\frac{1}{2}}\,(T^* - T_0)\left[\left\{\chi_0'(\xi) + \epsilon\,\chi_1'(\xi)\,e^{i\tau}\right\} + \right.$$
$$\left. + \frac{r^2\,\Omega}{\nu}\left\{(\phi_0'(\xi) + \epsilon\,\phi_1'(\xi)\,e^{i\tau}\right\}\right] \tag{47}$$

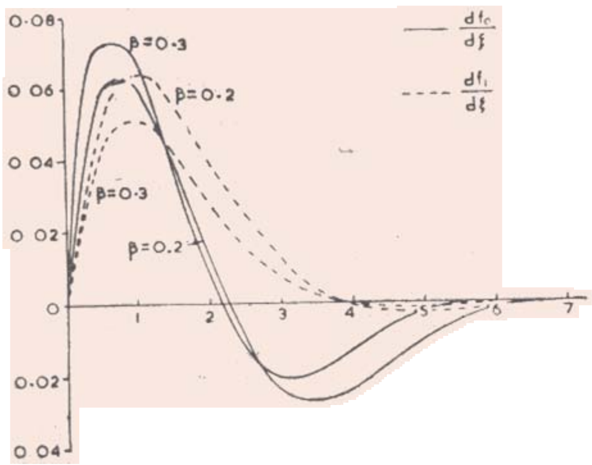

Fig. 1 — Variations of $\frac{df_0}{d\xi}$ and $\frac{df_1}{d\xi}$ against $\xi = z(\Omega/\nu)^{\frac{1}{2}}$.

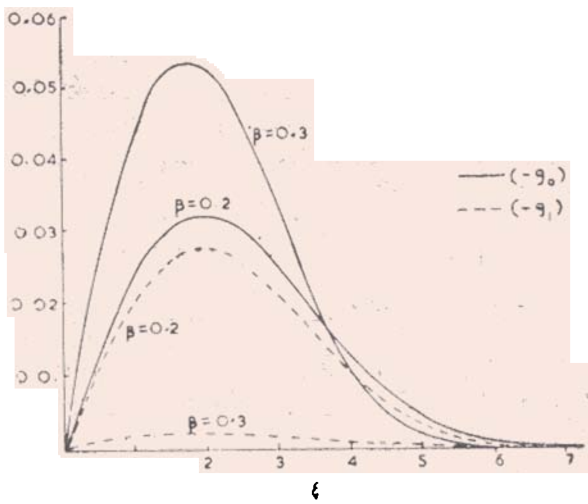

Fig. 2 — Variations of $-g_0$ and $-g_1$ against ξ.

where k is the thermal conductivity of the fluid. The oscillatory part of the heat transfer per unit area is given by

$$q_{osc} = \frac{k \epsilon}{D} (\Omega/\nu)^{\frac{1}{2}} (T^* - T_0) \left[\left\{ (A_2 + SB_2) + i \lambda (A_3 + S B_3) \right\} + \frac{r^2 \Omega}{\nu} \left\{ (E_0 - 4) + i \lambda E_1 \right\} \right] e^{i\tau} \quad (48)$$

The oscillatory part of the nondimensional rate of heat transfer from the fluid to the portion of the disk having radius $r = R (\nu/\Omega)^{\frac{1}{2}}$ is given by

$$\left[Q_{osc} \right]_r = \left[\int_0^r \left(q_{osc} \right) 2 \pi r \, d r \right] \Big/ \left[\pi (\nu/\Omega)^{3/2} (T^* - T_0) k \right] = M_R \cos (\tau - \Phi)$$

where

$$M_R = \frac{\epsilon R^2}{D} \left[\left\{ (A_2 + S B_2) + \tfrac{1}{2} R^2 (E_0 - 4) \right\}^2 + \lambda^2 \left\{ (A_3 + S B_3) + \tfrac{1}{2} R^2 E_1 \right\}^2 \right]^{\frac{1}{2}}$$

$$\tan \Phi = - \left[\lambda \left\{ (A_3 + S B_3) + \tfrac{1}{2} R^2 E_1 \right\} \Big/ \left\{ (A_2 + S B_2) + \tfrac{1}{2} R^2 (E_0 - 4) \right\} \right]$$

It is seen from (48) that for $R < R_1$ where $R_1^2 = (A_2 + S B_2)/4 E_0$, q_{osc} has a phase lag and if R exceeds R_1, this phase lag jumps to a phase lead, i.e., for $R > R_1$ heat is transferred from the disk to the fluid which will lead to the thermal instability. This shows that the solution holds good in the region $R \leqslant R_1$. The values of R_1 are given by $R_1 = 13 \cdot 134, 12 \cdot 951, 12 \cdot 758$, for $\beta = 0 \cdot 1, 0 \cdot 2, 0 \cdot 3$ respectively. In all the calculations we have taken $S = 1000$. Taking $\lambda \equiv 0 \cdot 1$, the values of Φ for $R = 5, 8, 10, 12$ and $\beta = 0 \cdot 1, 0 \cdot 2, 0 \cdot 3$ have been given in a bivariate Table 2. It is seen that Φ increases with the increase of R and decreases with the increase of β. Taking $\epsilon = 0 \cdot 1$ and $\lambda = 0 \cdot 1$, the values of M_R against R have been plotted in Fig. 3 for various values of β. It is seen that M_R increases with the increases of R as well as with the increase of β.

TABLE 2

BIVARIATE TABLE OF Φ FOR VARIOUS VALUES OF R AND β (IN DEGREES)

R	$\beta = 0 \cdot 1$	$\beta = 0 \cdot 2$	$\beta = 0 \cdot 3$
5	15 ·6950	7 ·8569	6 ·9557
8	18 ·1914	8 ·6986	8 ·2391
10	21 ·0791	10 ·3699	9 ·7976
12	25 ·7802	12 ·2976	12 ·1562

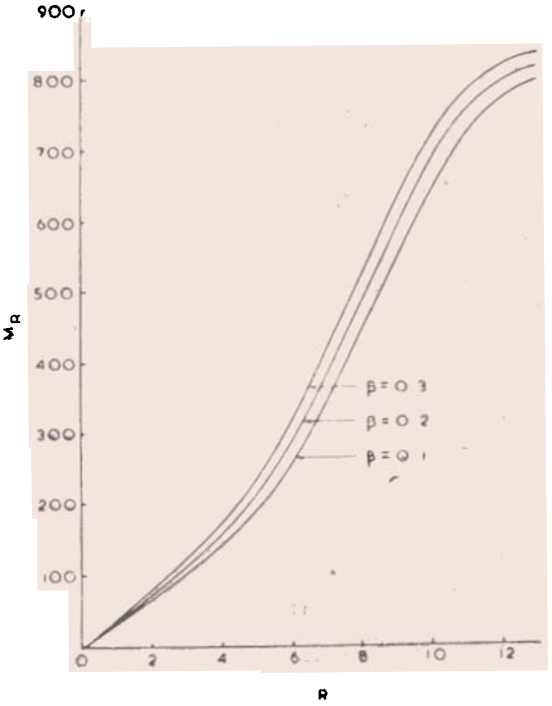

Fig. 3—Variations of the amplitude of oscillations of the heat transfer against dimensionless radius $R = r (\Omega/\nu)^{1/2}$.

ACKNOWLEDGEMENT

I am thankful to Prof. A.C. Srivastava, for his valuable help during the preparation of this paper.

REFERENCES

1. SRIVASTAVA, A. C. & BHATTACHARJEE, R. C., *Journal of Math. and Phy. Sciences.* IX (1975), 333–345.
2. LIGHTHILL, M. J., *Proc. Roy. Soc.*, A 224 (1954), 1–123.