A PROOF OF THE PUISEUX—HALPHEN INEQUALITY IN THE THEORY OF SPHERICAL PENDULUM BASED ON AN INTERESTING IDENTITY OF S. RAMANUJAN*

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The classical problem of finding bounds for the apsidal angle in the case of the spherical pendulum has been considered by Halphen in his classical treatise on '*Fonctions Elliptiques*'. His proof is based on certain inequalities among the Elliptic-function constant. We prove these inequalities of Halphen and Puiseux, in a simple way, using an interesting identity of S. Ramanujan; besides, we obtain positive-term series for the quantities involved which may enable one to improve such bounds.

In this paper we have established a connection between an interesting identity of S. Ramanujan and the famous classical inequalities of Puiseux and Halphen concerning the oscillatory motion of a Spherical pendulum. These inequalities relate to the problem of finding bounds for apsidal angle[1]. The proof given by Halphen[2] is based on the following inequalities among elliptic-function-constants, when one of the periods $2\omega_1$ is real and positive and the other purely imaginary $(0 < q < 1)$:

If

$$f(e_r) = e_r^2 - \eta_1 e_r/\omega_1 - g_2/6 , \quad r = 1, 2, 3. \tag{1}$$

then

$$f(e_1) < 0 , f(e_2) > 0 , f(e_3) < 0 , \tag{2}$$

where e_r $(r = 1, 2, 3)$, g_2 and η_1 are constants associated with Weierstrassian elliptic† and Zeta functions[3]; $P(\omega_1) = e_1$, $P(\omega_2) = e_2$, $P(\omega_1 + \omega_2) = e_3$, $\zeta(\omega_1) = \eta_1$.

The identity of Ramanujan[4] from which we derive the inequalities (2) is $(x = e^{2\pi i\, \omega_2/\omega_1})$:

$$\left(\frac{1}{4}\cot\frac{1}{2}\theta + \sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sin n\theta\right)^2 = \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{x^n\cos n\theta}{(1-x^n)^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{n\,x^n}{1-x^n}(1-\cos n\theta) \tag{3}$$

Written in terms of Weierstrassian functions, the identity (3) assumes the form[5]

$$\left(\zeta(u) - \frac{\eta_1 u}{\omega_1}\right)^2 - P(u) = \left(\frac{2\pi}{\omega_1}\right)^2\left[-\frac{1}{24} + \sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\cos\frac{n\pi u}{\omega_1}\right] \tag{4}$$

where

$$P(u) = \left(\frac{2\pi}{\omega_1}\right)^2\left[\frac{1}{24} + \frac{1}{4}\left(\cot\frac{\pi u}{2\omega_1}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi u}{\omega_1}\right]$$

and (5).

$$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1} .$$

*The material presented in this paper is included in the Ph.D. thesis of the author.

†Throughout the paper P denotes the Weierstrassian elliptic function.

By means of the identity (3) Ramanujan derives an interesting relation (identity (1) of Table 1)[4] which is equivalent to

$$\frac{\eta_1^2}{\omega_1^2} = \left(\frac{\pi}{\omega_1}\right)^4 \left[\frac{1}{144} + \frac{5}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}} - 2\sum_{n=1}^{\infty}\frac{n^2 q^{2n}}{(1-q^{2n})^2}\right]. \tag{6}$$

This identity is originally due to V.A. Lebesgue and has been used by Halphen to give explicit solution in terms of elliptic constants to the system of differential equations

$$\frac{d}{dt}(x_1 + x_2) = x_1 x_2 , \quad \frac{d}{dt}(x_2 + x_3) = x_2 x_3 , \quad \frac{d}{dt}(x_3 + x_1) = x_3 x_1,$$

which arose in connection with problems in mechanics and differential geometry.

In this paper we use the Ramanujan identity (4) to obtain the following interesting Lambert series for $f(e_r)$, $(r = 1, 2, 3)$:

$$\left.\begin{aligned} f(e_1) &= -2\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2} \\ f(e_2) &= \frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{n^2 q^n}{(1+q^n)^2} \\ f(e_3) &= -\frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{(2n-1)^2 q^{2n-1}}{(1+q^{2n-1})^2}, \end{aligned}\right\} \tag{7}$$

$o < q < 1$ and ω_1 are real and positive.

These relations evidently imply (2), and establishes bounds for the apsidal angle in the case of spherical pendulum and other related problems concerning the motion of a rigid body with one point fixed (Non-integrable case).

We need the Lambert series of e_r^2 and $\eta_1 e_r$ $(r = 1, 2, 3)$ for the proof of (7) ; series expansions of e_r^2 are well known and can be easily got from the relation

$$P''(\omega_r) = 6 e_r^2 - g_2/2 , \quad r = 1, 2, 3 . \tag{8}$$

Now[4]

$$g_2 = \frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4 \left(1 + 240\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}}\right). \tag{9}$$

Hence, we obtain

$$e_1^2 = \left(\frac{\pi}{2\omega_1}\right)^4 \left[\frac{4}{9} + \frac{80}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{(-1)^n n^3 q^{2n}}{1-q^{2n}}\right]. \tag{10}$$

$$e_2^2 = \left(\frac{\pi}{2\omega_1}\right)^4 \left[\frac{1}{9} + \frac{80}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^{2n}}\right] \tag{11}$$

$$e_3^2 = \left(\frac{\pi}{2\omega_1}\right)^4 \left[\frac{1}{9} + \frac{80}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}} + \frac{16}{3}\sum_{n=1}^{\infty}\frac{(-1)^n n^3 q^n}{1-q^{2n}}\right]. \tag{12}$$

(In deriving these relations we have used the Fourier series expansions of $P''(\omega_r)$ $(r = 1, 2, 3)$, which are obtained by differentiating (5) twice w.r.t. u and putting $u = \omega_r$).

The Lambert series expansions of $\eta_1 e_r$ $(r = 1, 2, 3)$, which are required in this connection are obtained by the use of the Ramanujan identity (4) in the following way; we prove :

$$\frac{\eta_1 e_1}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^4 \left[\frac{1}{72} + 2\sum_{n=1}^{\infty} \frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2} - \frac{5}{3}\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}} + \right.$$

$$\left. + \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^n n^3 q^{2n}}{1-q^{2n}}\right]. \tag{13}$$

$$\frac{\eta_1 e_2}{\omega_1} = \left(\frac{\pi}{\omega_1}\right)^4 \left[-\frac{1}{144} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^{2n})^2} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1-q^{2n})^2} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^{2n}} - \frac{5}{3}\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}}\right]. \tag{14}$$

These are believed to be new.

Proof of Eq. (13) : Changing u to $u + \omega_1$ in (4), we have

$$\left(\zeta(u+\omega_1) - \eta_1 - \frac{\eta_1 u}{\omega_1}\right)^2 - P(u+\omega_1) =$$

$$= \left(\frac{2\pi}{\omega_1}\right)^2 \left[-\frac{1}{24} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{(1-q^{2n})^2} \cos\frac{n\pi u}{\omega_1}\right]. \tag{15}$$

Taylor series expansion at $u=0$ of the left side of (15) is

$$\left[-u\left\{P(\omega_1) + \frac{\eta_1}{\omega_1}\right\} - \frac{u^3}{3!} P^{(2)}(\omega_1) - \frac{u^5}{5!} P^{(4)}(\omega_1) - \cdots\right]^2 - P(\omega_1) -$$

$$- \frac{u^2}{2} P^{(2)}(\omega_1) - \frac{u^4}{4!} P^{(4)}(\omega_1) - \cdots,$$

($P(r)$ denotes the (r)th derivative of P w.r.t. u) and that of right side is

$$\left(2\pi/\omega_1\right)^2 \left[-\frac{1}{24} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{(1-q^{2n})^2} \left(1 - \frac{1}{2}\left(\frac{n\pi u}{\omega_1}\right)^2 + \frac{1}{4!}\left(\frac{n\pi u}{\omega_1}\right)^4 - + \cdots\right)\right].$$

Hence, on comparing the coefficient of u^2 on either side of (15), we obtain

$$\left(P(u) + \frac{\eta_1}{\omega_1}\right)^2 - \frac{1}{2} P^{(2)}(\omega_1) = 2\,(\pi/\omega_1)^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 q^{2n}}{(1-q^{2n})^2}.$$

or we have

$$\left(e_1 + \frac{\eta_1}{\omega_1}\right)^2 - \frac{1}{2}\left(6e_1^2 - \frac{g_2}{2}\right) = 2\,(\pi/\omega_1)^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 q^{2n}}{(1-q^{2n})^2}.$$

Therefore

$$\frac{\eta_1 e_1}{\omega_1} = e_1^2 - \frac{g_2}{8} - \eta_1^2/2\omega_1^2 + \left(\pi/\omega_1\right)^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n^2 q^{2n}}{(1-q^{2n})^2}.$$

Substituting the series expansions of e_1^2, g_2 and η_1^2 from (10), (9) and (6) respectively in the last relation we get (13).

Proof of Eq. (14) : Changing u to $u + \omega_1$ in the identity (4) we get

$$\left(\zeta(u+\omega_2) - \eta_2 - \frac{\eta_1 u}{\omega_1} - \frac{\pi}{\omega_1}\right)^2 - P(u+\omega_2) =$$

$$= \left(\frac{2\pi}{\omega_1}\right)^2 \left[-\frac{1}{24} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{q^{3n}}{(1-q^{2n})^2} e^{n\pi u i/\omega_1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{q^n e^{-n\pi u i/\omega_1}}{(1-q^{2n})^2}\right]. \quad (16)$$

Comparing the coefficient of u^2 in the Taylor expansion of (16) at $u = 0$, we get

$$\left(P(\omega_2) + \frac{\eta_1}{\omega_1}\right)^2 - \frac{1}{2} P^{(2)}(\omega_2)$$

$$= -\left(\pi/\omega_1\right)^4 \left[\sum_{n=1}^{\infty} \frac{n^2 q^{3n}}{(1-q^{2n})^2} + \sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^{2n})^2}\right] \quad (17)$$

Since $$\sum_{n=1}^{\infty} \frac{n^2 q^{3n}}{(1-q^{2n})^2} = \sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^{2n})^2} - \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}},$$

the relation (17) gives

$$\frac{\eta_1 e_2}{\omega_1} = e_2^2 - \frac{g_2}{8} - \frac{\eta_1^2}{2\omega_1^2} + \left(\pi/\omega_1\right)^4 \left[\frac{1}{2}\sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^{2n})^2}\right].$$

Substituting the expansions (11), (9) and (6) for e_2^2, g_2 and η_1^2, respectively, in the last relation we obtain (14). The series expansion for $\eta_1 e_3$ can be easily got using the relations $\eta_1 e_3 = \eta_1 (e_1 + e_2)$.

We have from (1)

$$f(e_1) = e_1^2 - \eta_1 e_1/\omega_1 - g_2/6$$

Substituting the series expansions of e_1^2, $\eta_1 e_1/\omega_1$ and g_2 from (10), (13) and (6), respectively we get

$$\left(\omega_1/\pi\right)^4 f(e_1) = \frac{1}{36} + \frac{5}{3}\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}} + \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^n n^3 q^{2n}}{1-q^{2n}} -$$

$$-\frac{1}{72} - 2\sum_{n=1}^{\infty} \frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2} + \frac{5}{3}\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}} -$$

$$-\frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^n n^3 q^{2n}}{1-q^{2n}} - \frac{1}{72} - \frac{10}{3}\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}}.$$

Carrying out the evident simplification of the right hand side, we obtain finally

$$f(e_1) = -2\left(\pi/\omega_1\right)^4 \sum_{n=1}^{\infty} \frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2}.$$

This shows $f(e_1) < 0$ $(0 < q < 1$ and ω_1 positive). We notice that the last inequality holds for $0 > q > -1$ also. Again from (1)

$$f(e_2) = e_2^2 - \eta_1 e_2/\omega_1 - g_2/6.$$

Substituting the series for e_2^2, $\eta_1 e_2/\omega_1$ and g_2 from (11), (14) and (9), respectively, in the last relation we have

$$(\omega_1/\pi)^4 f(e_2) = \frac{1}{144} + \frac{5}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{(1-q^{2n})} + \frac{1}{3}\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^{2n}} + \frac{1}{144}$$

$$+ \frac{5}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{(1-q^{2n})} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2 q^n}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{n^2 q^n}{(1-q^{2n})^2}$$

$$-\sum_{n=1}^{\infty}\frac{n^2 q^{2n}}{(1-q^{2n})^2} - \frac{1}{3}\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^{2n}} - \frac{1}{72} - \frac{10}{3}\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1-q^{2n}}$$

$$= \sum_{n=1}^{\infty}\frac{n^2 q^n (1-q^n)}{(1-q^{2n})^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2 q^n}{1-q^{2n}}$$

$$= \sum_{n=1}^{\infty}\frac{n^2 q^n}{(1-q^{2n})(1+q^n)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2 q^n}{(1-q^{2n})}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2 q^n [2-(1+q^n)]}{(1-q^{2n})(1+q^n)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2 q^n}{(1+q^n)^2}.$$

or

$$f(e_2) = \frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{n^2 q^n}{(1+q^n)^2}.$$

This shows that $f(e_2) > 0$ $(0 < q < 1$ and ω_1 positive).

To obtain the series expansion for $f(e_3)$ in (7), we notice that $f(e_1) + f(e_2) + f(e_3) = 0$ [for $e_1 + e_2 + e_3 = 0$ & $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$].

Therefore

$$f(e_3) = -f(e_1) - f(e_2)$$

$$= 2 \cdot \frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2} - \frac{1}{2}\left(\frac{\pi}{\omega_1}\right)^4 \sum_{n=1}^{\infty}\frac{n^2 q^n}{(1+q^n)^2}.$$

Now

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2 q^{4n-2}}{(1-q^{4n-2})^2} = \sum_{n=1}^{\infty}\frac{n^2 q^{2n}}{(1-q^{2n})^2} - \sum_{n=1}^{\infty}\frac{(2n)^2 q^{4n}}{(1-q^{4n})^2}$$

$$= \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1-q^{2n})^2} - \left(\sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1-q^{2n})^2} - \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1+q^{2n})^2} \right)$$

$$= \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{(1+q^{2n})^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(2n)^2 q^{2n}}{(1+q^{2n})^2} .$$

Hence, we have

$$f(e_3) = - \frac{1}{2} \left(\pi/\omega_1 \right)^4 \sum_{n=1}^{\infty} \frac{(2n-1)^2 q^{2n-1}}{(1+q^{2n-1})^2}, \qquad (18)$$

hence it follows

$$f(e_3) < 0 \quad (0 < q < 1, \ \omega_1 > 0).$$

We have obtained here positive term series in place of the inequalities used by Halphen[2]; this may enable us to obtain an improvement on the Puiseux-Halphen bounds in the case of the spherical pendulum and other related problems.

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*From these relations it follows that

$$4f(e_r) = - \left(\pi/\omega_1 \right)^2 \quad q \ \frac{de}{dq} \quad (r = 1, 2.$$