THE NUMBER SERVED IN A HEAD-OF-THE-LINE PRIORITY QUEUE WITH GENERAL SERVICE*

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In this paper, using the supplementary variable method, the head-of-the-line priority queueing system with general service time distributions has been studied to obtain an expression for the Laplace transform of the generating function of the joint probability distribution of the number of priority and non-priority units in the queue at time t, and the number of units served (including non-priority units) in time t. In particular cases, explicit solution for the model with exponential service time distributions have been obtained; expression for the joint distribution of the queue length and the number served in time t, for an M/G/1 queue has also been derived.

The head-of-the-line priority discipline seems to have been introduced by Cobham[1]. Jaiswal[2] studied the head-of-the line priority queueing system by using the supplementary variable method and obtained the generating functions of the joint distribution of queue length for the first and second priority units. In the analysis of the above mentioned models, authors have confined their attention to the queue length; from the management point of view, we can extract more information about the system behaviour by making more comprehensive description of the state of the system, for example, by the inclusion of the number of units served at any time t.

In the present paper, by including the above mentioned additional information, we study the head-of-the-line priority queue discipline with Poisson arrivals and general service time distributions. Using the supplementary variable technique, we obtain an expression for the Laplace transform of the generating function of the joint probability distribution of the number of priority and non-priority units in the queue at a time t, and the number of units served (including non-priority) at the time t. In particular cases, the same model with exponential service time distributions has been studied and complete solution obtained; correspondence with earlier results of Gaur[3] and Jaiswal[2] have been shown.

It is to be noted here that some work on this line of action for a non-priority discipline have already been done by Prabhu[4], Prabhu and Bhat[5], Greenberg and Greenberg[6], Scot[7] and Guar[3].

STATEMENT OF THE PROBLEM

At a service facility, the priority and non-priority units arrive according to Poisson distributions with λ_1 and λ_2 as their respective mean arrival rates. There is a single server and selection for service is according to the head-of-the-line priority discipline. The service time distributions of both types are general with probability densities $S_1(x)$ and $S_2(x)$ respectively. Let $\eta_1(x) \triangle$ be the first order probability that a priority unit completes service in the interval $(x, x + \triangle)$, if the unit has already been in service for a time x, $\eta_2(x) \triangle$ having a similar meaning for the non-priority units, so that

$$S_1(x) = \eta_1(x) \exp \left[- \int_0^x \eta_1(u)\, du \right]$$

and

$$S_2(x) = \eta_2(x) \exp \left[- \int_0^x \eta_2(u)\, du \right]$$

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Let us define the following probabilities :

(a) $P_{m,n,k}(x,t)\,dx$ $(m \geqslant 1,\ n \geqslant 0,\ k \geqslant 0)$ is the probability that at time t there are m priority and non-priority units in the system and k units (including non-priority) have been served in time t, and a priority unit is in service with elapsed service time lying between x and $x + dx$;

(b) $Q_{m,n,k}(x,t)\,dx$ $(m \geqslant 0,\ n \geqslant 1,\ k \geqslant 0)$ is the probability that at time t there are m priority and n non-priority units in the system, and k units have been served in time t, and a nor-priority unit is in service with elapsed service time lying between x and $x + dx$;

(c) $P_{o,k}(t)$ $(k \geqslant 0)$ is the probability that at time t there is neither a priority nor a non-priority unit in the system, while k units (including non-priority) have been served in time t.

FORMULATION OF EQUATION

Following Keilson and Kooharian[8], we derive the difference-differential equations for the system :

$$\frac{\partial}{\partial t} P_{m,n,k}(x,t) + \frac{\partial}{\partial x} P_{m,n,k}(x,t) + \{\lambda_1 + \lambda_2 + \eta_1(x)\} P_{m,n,k}(x,t) = = \lambda_1 P_{m-1,n,k}(x,t) + \lambda_2 P_{m,n-1,k}(x,t) \quad (m \geqslant 1,\ n \geqslant 0,\ k \geqslant 0) \tag{1}$$

$$\frac{\partial}{\partial t} Q_{m,n,k}(x,t) + \frac{\partial}{\partial x} Q_{m,n,k}(x,t) + \{\lambda_1 + \lambda_2 + \eta_2(x)\} Q_{m,n,k}(x,t) = = \lambda_1 Q_{m-1,n,k}(x,t) + \lambda_2 Q_{m,n1,k}(x,t), \quad (m \geqslant 0,\ n \geqslant 1,\ k \geqslant 0) \tag{2}$$

$$\frac{d}{dt} P_{0,k}(t) + \{\lambda_1 + \lambda_2\} P_{0,k}(t) = \int_0^\infty P_{1,0,k-1}(t)\,dx\,\eta_1(x)\,dx + \int_0^\infty Q_{0,1,k-1}(x,t)\,\eta_2(x)\,dx, \tag{3}$$

where $P_{m,n,k}(x,t)$, $Q_{m,n,k}(x,t)$ vanish for m or $n < 0$. These equations are to be solved under the following boundary conditions :

$$P_{m,n,k}(0,t) = \int_0^\infty P_{m+1,n,k-1}(x,t)\,\eta_1(x)\,dx + \int_0^\infty Q_{m,n+1,k-1}(x,t)\,\eta_2(x)\,dx, \quad (m \geqslant 2,\ n \geqslant 1,\ k \geqslant 0) \tag{4}$$

$$P_{1,0,k}(0,t) = \int_0^\infty P_{2,0,k-1}(x,t)\eta_1(x)\,dx + \int_0^\infty Q_{1,1,k-1}(x,t)\,\eta_2(x)\,dx + \lambda_1 P_{0,k}(t) \tag{5}$$

$$Q_{0,n,k}(0,t) = \int_0^\infty Q_{0,n+1,k-1}(x,t)\,\eta_2\,dx + \int_0^\infty P_{1,n,k-1}(x,t)\,\eta_1(x)\,dx \tag{6}$$

$$Q_{0,1,k}(0,t) = \int_0^\infty Q_{0,2k-1}(x,t)\,\eta_2(x)\,dx + \int_0^\infty P_{1,1,k-1}(x,t)\,\eta_1(x)\,dx + \lambda_2 P_{0,k}(t) \tag{7}$$

and the initial condition $P_{0,0}(0) = 1$ (i.e. the system starts from an empty state).

SOLUTION OF THE PROBLEM

We now define the following generating functions :

$$f_{m,k}(x,\alpha,t) = \sum_{n=0}^{\infty} \alpha^n P_{m,n,k}(x,t)\,;\ f_k(x,\alpha,\beta,t) = \sum_{m=0}^{\infty} f_{m,k}(x,\alpha,t)\,\beta^m$$

$$F(x,\alpha,\beta,\delta,t) = \sum_{k=0}^{\infty} \delta^k f_k(x,\alpha,\beta,t);$$

$$g_{m,k}(x,\alpha,t) = \sum_{n=1}^{\infty} \alpha^n Q_{m,n,k}(x,t) \; ; \; g_k(x,\alpha,\beta,t) \sum_{m=0}^{\infty} \beta^m g_{m,k}(x,\alpha,\beta,t)$$

$$G(x,\alpha,\beta,\delta,t) = \sum_{k=0}^{\infty} \delta^k g_k(x,\alpha,\beta,t) \; ;$$

$$H_m(x,\alpha,\delta,t) = \sum_{k=0}^{\infty} \delta^k f_{m,k}(x,\alpha,t) \; ; \; I_m(x,\alpha,\delta,t) = \sum_{k=0}^{\infty} \delta^k g_{m,k}(x,\alpha,\beta), \; J_0(\delta,t) = \sum_{k=0}^{\infty} \delta^k P_{0,k}(t).$$

Also, we define Laplace transform of a function $F(t)$ by

$$\overline{F}(s) = \int_0^{\infty} e^{-st} F(t)\, dt.$$

Now, we multiply eqns. (1) to (7) by appropriate powers of α,β,δ and using the generating functions defined above, and then applying Laplace transform, we have the solutions

$$\overline{F}(x,\alpha,\beta,\delta,s) = A_1(\alpha,\beta,\delta,s) \exp\left[-\{s+\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x - \int_0^x \eta_1(u)\,du\right] \tag{8}$$

$$\overline{G}(x,\alpha,\beta,\delta,s) = A_2(\alpha,\beta,\delta,s) \exp\left[-\{s+\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x - \int_0^x \eta_2(u)\,du\right] \tag{9}$$

$$\overline{F}(0,\alpha,\beta,\delta,s) = \frac{\delta}{\beta}\int_0^{\infty} \overline{F}(x,\alpha,\beta,\delta,s)\,\eta_1(x)\,\mathrm{dx} + \lambda_1 \beta \overline{J}_0(\delta,s) + \frac{\delta}{\alpha}\int_0^{\infty} \overline{G}(x,\alpha,\beta,\delta,s)\,\eta_2(x)\,dx -$$

$$- \delta \int_0^{\infty} \overline{H}_1(x,\alpha,\delta,s)\,\eta_1(x)\,dx - \frac{\delta}{\alpha}\int_0^{\infty} \overline{I}_0(x,\alpha,\delta,s)\,\eta_2(x)\,dx, \tag{10}$$

$$\overline{I}_0(0,\alpha,\delta,s) = \frac{\delta}{\alpha}\int_0^{\infty} \overline{I}_0(x,\alpha,\delta,s)\,\eta_2(x)\,dx + \delta\int_0^{\infty} \overline{H}_1(x,\alpha,\delta,s)\,\eta_1(x)\,dx -$$

$$- \{s+\lambda_1+\lambda_2(1-\alpha)\}\,\overline{J}_0(\delta,s) + 1. \tag{11}$$

Since a non-priority unit is taken for service, if and only if, there are no priority units in the system. The expression $Q_{m,n,k}(x,t)$ must vanish for $x=0$, $m>0$, $k>0$, and $Q_{0,n,k}(0,t)$ for $x=0$, so that $\overline{I}_m(0,\alpha,\delta,s) = 0$, $m>0$ and $\overline{I}_0(0,\alpha,\delta,s) \neq 0$.

Hence

$$\overline{G}(0,\alpha,\beta,\delta,s) = \overline{I}_0(0,\alpha,\delta,s) = \overline{G}(0,\alpha,0,\delta,s),$$

so that

$$A_2(\alpha,\beta,\delta,s) = A_2(\alpha,0,\delta,s) = A_2(\alpha,\delta,s) \text{ (say)}.$$

Then, we have

$$\overline{G}(x,\alpha,\beta,\delta,s) = A_2(\alpha,\delta,s) \exp\left[-\{s+\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}x - \int_0^x \eta_2(u)\,du\right] \tag{12}$$

$$\overline{I_0}(x,\alpha,\delta,s) = A_2(\alpha,\delta,s)\exp\left[-\{s+\lambda_1+\lambda_2(1-\alpha)\}x - \int_0^x \eta_2(u)\,du\right] \tag{13}$$

Substituting values of (12) and (13) in (16) and (11),

we have

$$A_1(\alpha,\beta,\delta,s)\left[1 - \frac{\delta}{\beta}\,\overline{S_1}\{\lambda_1(1-\beta)+\lambda_2(1-\beta)+s\}\right] =$$

$$= \lambda_1\,\beta\,\overline{J}_0(\delta,s) - \delta\int_0^\infty \overline{H}_1(x,\alpha,\delta,s)\,\eta_1(x)\,dx + \frac{\delta}{\alpha}\,A_2(\alpha,\delta,s)$$

$$\left[\overline{S_2}\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\} - \overline{S}_2\{\lambda_1+\lambda_2(1-\alpha)+s\}\right]. \tag{14}$$

$$A_2(\alpha,\delta,s)\left[1-\frac{\delta}{\alpha}\,\overline{S_2}\{\lambda_1+\lambda_2(1-\alpha)+s\}\right] = \delta\int_0^\infty \overline{H_1}(x,\alpha,\delta,s)\,\eta_1(x)\,dx -$$

$$\{s+\lambda_1+\lambda_2(1-\alpha)\}\,J_0\overline{(\delta,s)} + 1, \tag{15}$$

where $\overline{S_1}(s)$ and $\overline{S_2}(s)$ are the Laplace transforms of $S_1(x)$ and $S_2(x)$ respectively.

Now, it can be easily shown by Rouché's theorem that the equation $\beta - \delta\, S_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\} = 0$ has one and only one root inside the unit circle $|\beta| = 1$ for Re $s > 0$, $|\delta| < 1$, then following Ta- kács[2] the value of the root can be determined. Let $\beta = \beta(\alpha,s)$ be the required root. Then from eqn. (14), on putting $\beta = \beta(\alpha,s)$, we have

$$\delta\int_0^\infty \overline{H}_1(x,\alpha,\delta,s)\,\eta_1(x)\,dx = \lambda_1\,\beta(\alpha,s)\cdot\overline{J_0}(\delta,s) + A_2(\alpha,\delta,s)\,[\overline{S}_2\{\lambda_1(1-\beta)(\alpha,s)) +$$

$$+\lambda_2(1-\alpha)+s\} - \overline{S}_2\{s+\lambda_1+\lambda_2(1-\alpha)\}]. \tag{16}$$

From eqn. (15) and (16), we have

$$A_2(\alpha,\delta,s) = \frac{1-[s+\lambda_1\{1-\beta(\alpha,s)\}+\lambda_2(1-\alpha)]\,\overline{J_0}(\delta,s)}{1-\dfrac{\delta}{\alpha}\,\overline{S_2}[s+\lambda_1\{1-\beta(\alpha,s)\}+\lambda_2(1-\alpha)]} \tag{17}$$

Substituting for $\int_0^\infty \overline{H}_1(x,\delta,s)\,\eta_1(x)\,dx$ from (15) in (14), we have

$$A_1(\alpha,\beta,\delta,s) = \frac{1-\{s+\lambda_1(1-\beta)+\lambda_2(1-\alpha)\}\,\overline{J_0}(\delta,s)}{1-\dfrac{\delta}{\beta}\,\overline{S}_1\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}} -$$

$$-A_2(\alpha,\delta,s)\,\frac{\left[1-\dfrac{\delta}{\alpha}\,\overline{S_2}\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}\right]}{\left[1-\dfrac{\delta}{\beta}\,\overline{S_1}\{\lambda_1(1-\beta)+\lambda_2(1-\alpha)+s\}\right]}. \tag{18}$$

Thus, eqn. (8) and (9) are completely determined except for $\overline{J_0}(\delta,s)$, which we can determine by the usual argument.

By Rouché's theorem it can be shown that denominator of eqn. (17) has one and only one zero inside the unit circle $|\alpha| < 1$ for Re $s > 0$, $|\delta| < 1$, and at this zero, the numerator should also vanish, so that $A_2(\alpha,\delta,s)$ may be regular inside the unit circle. Let α_s be that zero, then

$$\overline{J}_0(\delta,s) = \frac{1}{[s+\lambda_1\{1-\beta(\alpha_s,s)\}+\lambda_2(1-\alpha_s)]} \tag{19}$$

Hence, if $\pi\ (\alpha, \beta, \delta, t)$ represents the generating function of the joint probability distribution of the number of priority and non-priority units in the queue at time t, and the number of units already served at time t, its Laplace transform is given by

$$\bar{\pi}(\alpha, \beta, \delta, s) = \int_0^\infty \bar{F}(x, \alpha, \beta, \delta, s)\, dx + \int_0^\infty \bar{G}(x, \alpha, \beta, \delta, s)\, dx + \bar{J}_0(\delta, s)$$

$$= A_1(\alpha, \beta, \delta, s) \frac{\left[1 - \bar{S}_1\{\lambda_1(1-\beta) + \lambda_2(1-\alpha) + s\right]}{\left[\lambda_1(1-\beta) + \lambda_2(1-\alpha) + s\right]} +$$

$$+ A_2(\alpha, \beta, \delta, s) \frac{\left[1 - \bar{S}_2\{\lambda_1(1-\beta) + \lambda_2(1-\alpha) + s\}\right]}{\left[\lambda_1(1-\beta) = \lambda_2(1-\alpha) + s\right]} + \bar{J}_0(\delta, s) \quad (20)$$

where $A_1(\alpha, \beta, \delta, s)$, $A_2(\alpha, \beta, \delta, s)$ and $\bar{J}_0(\delta, s)$ are given by eqn. (17), (18) and (19).

Particular Cases

(*i*) *Exponential service distribution*:—In this section, attempt has been made to illustrate the procedure by which complete solution of the problem can be obtained; when the service times of priority and non-priority units follow exponential distributions viz. $S_1(t) = \mu_1\, e^{-\mu_1}$ and $S_2(t) = \mu_2\, e^{-\mu_2 t}$. In the formulation of equations, now, x will be dropped and $\eta_1(x)$, $\eta_2(x)$ are replaced by μ_1, μ_2, while $P_{m,n,k}(x, t)$, $Q_{m,n,k}(x, t)$ by $P_{m,n,k}(t)$ and $Q_{m,n,k}(t)$ respectively.

Then equations (8), (9), (10) and (11) give us

$$\left\{s + \lambda_1(1-\beta) + \lambda_2(1-\alpha) + \mu_1\left(1 - \frac{\delta}{\beta}\right)\right\} \bar{F}(\alpha, \beta, \delta, s) = \lambda_1 \beta\, \bar{J}_0(\delta, s) +$$

$$+ \frac{\mu_2 \delta}{\alpha} \bar{G}(\alpha, \beta, \delta, s - \mu_1 \delta\, \bar{H}_1(\alpha, \delta, s) - \frac{\mu_2 \delta}{\alpha} \bar{I}_0(\alpha, \delta, s). \quad (21)$$

$$\left\{s + \lambda_1(1-\beta) + \lambda_2(1-\alpha) + \mu_2\right\} \bar{G}(\alpha, \beta, \delta, s) = 1 - \left\{s + \lambda_1 + \lambda_2(1-\alpha)\right\} \bar{J}_0(\delta, s) +$$

$$+ \mu_1 \delta\, \bar{H}_1(\alpha, \delta, s) - \left\{\lambda_2 \alpha - \frac{\mu_2 \delta}{\alpha}\right\} \bar{I}_0(\alpha, \delta, s). \quad (22)$$

$$\left\{s + \lambda_1 + \mu_2\left(1 - \frac{\delta}{\alpha}\right)\right\} \bar{I}_0(\alpha, \delta, s) = 1 - \left\{s + \lambda_1 + \lambda_2(1-\alpha)\right\} \bar{J}_0(\delta, s) +$$

$$+ \mu_1 \delta\, \bar{H}_1(\alpha, \delta, s). \quad (23)$$

Now eqn. (22) with the help of (23) becomes

$$\left\{s + \lambda_1(1-\beta) + \lambda_2(1-\alpha) + \mu_2\right\} \bar{G}(\alpha, \beta, \delta, s) = \left\{s + \lambda_2(1-\alpha) + \lambda_1 + \mu_2\right\} \bar{I}_0(\alpha, \delta, s) \quad (24)$$

Substituting values of $\overline{G}(\alpha, \beta, \delta, s)$ and $\mu_1 \delta \overline{H}_1(\alpha, \delta, s)$ from eqn. (24) and (23) in eqn. (21) and simplifying we have

$$\left\{ s + \lambda_1 (1-\beta) + \lambda_2 (1-\alpha) + \mu_1 \left(1 - \frac{\delta}{\beta}\right) \right\} \overline{F}(\alpha, \beta, \delta, s) = 1 - \left\{ s + \lambda_1 (1-\beta) + \right.$$

$$\left. + \lambda_2 (1-\alpha) \right\} \overline{J_0}(\delta, s) - \frac{\mu_2 \delta}{\alpha} \left[1 - \left\{ \frac{s + \lambda_2 (1-\alpha) + \lambda_1 + \mu_2}{s + \lambda_1(1-\beta) + \lambda_2 (1-\alpha) + \mu_2} \right. \right.$$

$$\left. \left. + \frac{\alpha}{\mu_2 \delta} \left(s + \lambda_1 + \lambda_2 + \mu_2 (1 - \frac{\delta}{\alpha}) \right) \right\} \right] \times \overline{I_0}(\alpha, \delta, s). \qquad (25)$$

Now denominator of eqn. (25) is a quadratic equation in β; let β_1 be one of the roots lying inside the unit circle $|\beta| < 1$ and is given by

$$\beta_1 = \frac{(s + \lambda_1 + \lambda_2 (1-\alpha) + \mu_1) - \left[\left\{ s + \lambda_1 + \lambda_2 (1-\alpha) + \mu_1 \right\}^2 - 4\lambda_1 \mu_1 \delta \right]^{\frac{1}{2}}}{2\lambda_1}.$$

Then by the usual arguments of Rouché's theorem, we have

$$\overline{I_0}(\alpha, \delta, s) = \frac{1 - \{ s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha) \} \overline{J_0}(\delta, s)}{\frac{\mu_2 \delta}{\alpha} \left[\frac{-\lambda_1 \beta_1}{s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha) + \mu_2} - \frac{\alpha}{\mu_2 \delta} \left(s + \lambda_1 + \lambda_2 + \mu_2 (1 - \frac{\delta}{\alpha}) \right) \right]}. \qquad (26)$$

Again denominator of (26) is a quadratic equation in α,

Viz $A\alpha^2 - B\alpha + C = 0$

Where

$$A = \lambda_2 (s + \lambda_1 + \lambda_2 + \mu_2)$$
$$B = -\{ \lambda_2 \delta + (s + \lambda_1 + \lambda_2 + \mu_2)(s + \lambda_2 + \mu_2 + \lambda_1 (1-\beta_1)) \}$$
$$C = \delta \{ s + \lambda_2 + \lambda_1(1-\beta_1) + \mu_2 (1 - \lambda_1 \beta_1) \},$$

using the same arguments let $\alpha = \alpha_1$ be that root which lies inside the unit circle $|\alpha| < 1$. Then we have

$$\overline{J}_0(\delta, s) = \frac{1}{[s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha_1)]}, \qquad (27)$$

where

$$\alpha_1 = \frac{B - (B^2 - 4AC)^{\frac{1}{2}}}{2A}.$$

Thus once knowing β_1 and α_1, $\overline{J_0}(\delta, s)$ is completely determined. Now eqn. (26) with the help of (27) gives us

$$\overline{I}_0(\alpha, \delta, s) = \frac{\left[1 - \frac{(s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha)}{(s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha_1)} \right]}{\left[\frac{-\lambda_1 \beta_1 \mu_2 \delta}{\alpha \{ s + \lambda_1 (1-\beta_1) + \lambda_2 (1-\alpha) + \mu_2 \}} - \left\{ s + \lambda_1 + \lambda_2 + \mu_2 \left(1 - \frac{\delta}{\alpha}\right) \right\} \right]} \qquad (28)$$

Further eqn. (24) with the help of (28) becomes

$$\overline{G}(\alpha, \beta, \delta, s) = \frac{\left[\dfrac{s + \lambda_2(1-\alpha) + \lambda_1 + \mu_2}{s + \lambda_1(1-\beta) + \lambda_2(1-\alpha) + \mu_2}\right]\left[1 - \left(\dfrac{s + \lambda_1(1-\beta_1) + \lambda_2(1-\alpha)}{s + \lambda_1(1-\beta_1) + \lambda_2(1-\alpha_1)}\right)\right]}{\left[\dfrac{-\lambda_1 \beta \mu_2 \delta}{\alpha\{s + \lambda_1(1-\beta_1) + \lambda_2(1-\alpha) + \mu_2\}} - \left\{s + \lambda_1 + \lambda_2 + \mu_2\left(1 - \dfrac{\delta}{\alpha}\right)\right\}\right]} \tag{29}$$

The value of $\overline{H}_1'(\alpha, \delta, s)$ can be evaluated from eqn. (22) or (23) with the help of eqns. (27), (28) and (29). Once knowing all these functions, value of $\overline{F}(\alpha,\beta,\delta,s)$ can be obtained from eqn. (21). Thus all these functions are explicitly determined which give complete solution of the problem. In the steady state cases, other parameters of operational characteristics can be computed by using the usual methods e.g.

$$\frac{\partial}{\partial \delta} J_0(\delta) \mid_{\delta=1} \quad = \text{Mean number of units served} \tag{30}$$

$$\frac{\partial}{\partial \alpha} I_0(\alpha, \delta) \mid_{\alpha=\delta=1} \quad = \text{Mean number of non-priority units in the system when there are no priority units waiting for service.} \tag{31}$$

$$\frac{\partial}{\partial \beta} G(\alpha, \beta, \delta) \mid_{\beta=\alpha=\delta=1} = \text{Mean number of priority units in the system when a non-priority unit is in service,} \tag{32}$$

etc. However, the complete solution in the steady-state case is to be submitted elsewhere for publication.

(*ii*) For an M|G|1 queue, we assume that all units are served as ordinary (non-priority) units, so that $P_{m,n,k}(x, t)$, λ_1 and $\eta_1(x)$ are all meaningless, so dropping them in eqn. (11) and (13) replacing λ_2, $\eta_2(x)$ by λ and $\eta(x)$ respectively, we have

$$\overline{I}_0(x, \alpha, \delta, s) = A_2(\alpha, \delta, s) \exp\left[\left\{s + \lambda(1-\alpha)\right\} x - \int_0^x \eta(u)\, du\right] \tag{33}$$

where

$$A_2(\alpha, \delta s) = \frac{\left[1 - \left\{s + \lambda(1-\alpha)\right\} \sum_{k=0}^{\infty} \overline{P}_{0,k}(S)\, \delta^k\right]}{\left[1 - \dfrac{\delta}{\alpha}\, \overline{S}\left\{s + \lambda\left(1 - a\right)\right\}\right]},$$

which is in fact eqn. (3.1, 47) of our earlier work (Gaur[3]) obtained slightly in a different form, due to the difference in the initial condition.

(*iii*) Eqn. (20) is in an agreement with eqn. 27 of Jaiswal[2], on dropping K and δ, since in the latter the number served is not considered.

Applications of the Model

Applications of the head-of-the-line priority queueing model of defence interest are encountered at an E.M.E. workshop. The repair and maintenance of A and B vehicles and several military equipments are carried out at various E.M.E. workshops. Generally service is performed on the basis of FIFO discipline but during operation time or emergency, priority in service is assigned to some of the equipment. It has been observed that the head-of-the-line priority rule is frequently used. Often, management is interested in knowing about the number of equipment served, joint probability of the number of equipment waiting and the number of equipment served, the average queue lengths of equipment waiting for service in different situations etc. As explained earlier, eqns. (27) to (32) are used to derive the above mentioned necessary information. This information is also quite useful in determining the inventories of spare parts since the demand for the replacement of spare parts is generated by the quipment coming for servicing.

Further, the problem studied can be considered as a 'breakdown model' in which the repair of a 'priority breakdown' can be postponed until the unit in service completes its service. Jain[10] has considered a similar model with priority resume repair policy.

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