PROPAGATION OF FREE TORSIONAL WAVES IN A NON-HOMOGENEOUS MAGNETO-VISCO-ELASTIC SLAB WITH A CYLINDRICAL HOLE

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The aim of the present paper is to investigate the propagation of free torsional waves in a non-homogeneous magneto-visco-elastic slab with a cylindrical hole and obtain frequency equation. The shear modulus μ and the density ρ of the slab are assumed to vary as some power of the radial distance.

Recently Chakravorti[1] discussed the propagation of torsional waves in a perfectly, conducting elastic cylinder and a tube under the influence of a uniform axial magnetic field. Chandrashekharaiah[2], by considering a circular cylinder and a circular tube, made up of a perfectly conducting visco-elastic material representing a parallel union of the Kelvin and Maxwell bodies, studied the propagation of torsional waves in these bodies in the presence of an axial magnetic field. Chakravarty[3] discussed the problem of free torsional vibrations of an inhomogeneous slab with a cylindrical hole. As a sequal to these, the present paper is an attempt to discuss the propagation of free torsional waves in a non-homogeneous magneto-visco-elastic slab with a cylindrical hole. We assume that the slab is made of a perfectly conducting non-homogeneous visco-elastic material representing a parallel union of Kelvin and Maxwell bodies. The non-homogeneity of the slab is due to the variable shear modulus μ and variable density ρ. Two cases have been considered : (a) when both the density ρ and shear modulus μ of the slab vary as some power of radial distance; (b) when ρ is constant and only μ varies. In both the cases frequency equations have been derived. Such problems of magneto-elastic-vibrations have been found to be useful in various branches of science like Astrophysics, Plasma-physics, Acoustics[4], etc.

PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Consider a perfectly conducting non-homogeneous visco-elastic slab with a cylindrical hole placed in a magnetic field and surrounded by vacuum. We suppose that the axis of z-coincides with the axis of the cylindrical hole. Referred to the cylindrical co-ordinates (r, θ, z) the faces of the slab are $z = \pm h$ and the cylindrical hole surface is given by $r=a$. The problem being one of magneto-visco-elasticity, the fundamental equations are those of electro-magnetism and of visco-elasticity. Maxwell's equations governing the electromagnetic field are,

$$\text{curl } \vec{H} = 4\pi \vec{J} \tag{1}$$

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\text{div } \vec{B} = 0 \tag{3}$$

$$\vec{B} = \mu_e \vec{H} \tag{4}$$

where the displacement current is neglected and Gaussian units have been used. By Ohm's law, we have,

$$J = \sigma\left[\vec{E} + \frac{1}{c}\frac{\partial \vec{u}}{\partial t} \times \vec{B}\right] \tag{5}$$

In equations (1) to (5) $\vec{H}, \vec{B}, \vec{E}, \vec{J}$ respectively denote the magnetic intensity, magnetic induction, electric intensity and current density vectors, μ_e and σ respectively denote magnetic permeability and electrical conductivity of the slab; $\vec{u}$ represents the displacement vector in the strained state and c is the velocity of light.

Assuming that the temperature remains constant, the stress-strain relation as given in Nowacki[5] for visco-elastic solid, is,

$$\left(1 + m_1 \frac{\partial}{\partial t}\right) S_{ij} = 2\mu \left(1 + m_2 \frac{\partial}{\partial t}\right) e_{ij} \tag{6}$$

where

$$S_{ij} = \sigma_{ij} - \frac{1}{3} s \, \delta_{ij}, \ (s = 3ke) \tag{7}$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} e \, \delta_{ij}, \ (e = \epsilon_{kk})$$

are deviatoric components of the stress and strain tensors σ_{ij} and ϵ_{ij}, λ and μ are Lame's constants, $K = \lambda + 2\mu/3$ is the bulk modulus, m_1, m_2 are visco-elastic moduli and δ_{ij} is Kronecker's delta. The strain displacement relation is

$$2\epsilon_{ij} = u_{ij} + u_{j,i} \tag{8}$$

and

$$\sigma_{ij,j} + (\vec{J} \times \vec{B})_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{9}$$

where ρ is the density of the slab. From equations (1) to (9) it is clear that the electromagnetic field is interaction with the mechanical field due to the presence of $\vec{u}$ in (5). Eliminating S_{ij}, e_{ij} and s from equations (6) and (7), we obtain,

$$\left(1 + m_1 \frac{\partial}{\partial t}\right) \sigma_{ij} = \left\{ \lambda \left(1 + m_1 \frac{\partial}{\partial t}\right) + \frac{2}{3} \mu (m_1 - m_2) \frac{\partial}{\partial t} \right\} e \delta_{ij} + 2\mu \left(1 + m_2 \frac{\partial}{\partial t}\right) \epsilon_{ij} \tag{10}$$

Moreover, the electromagnetic field equations in vacuum are,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{E}^* = 0 \tag{11}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{h}^* = 0 \tag{12}$$

$$\text{curl } \vec{E}^* = -\frac{1}{c} \frac{\partial \vec{h}^*}{\partial t} \tag{13}$$

$$\text{curl } \vec{h}^* = \frac{1}{c} \frac{\partial \vec{E}^*}{\partial t} \tag{14}$$

where $\vec{h}$ is the perturbation of the magnetic field and $\vec{E}^*$ is the electric field in vacuum.

Since we are considering torsional vibration, the displacement vector $\vec{u}$ has only v as its non-vanishing component which is independent of θ in cylindrical co-ordinates, i.e.,

$$u_r = u_z = 0 \tag{15}$$
$$u_\theta = v = f(r, z)\, e^{ipt}$$

and the magnetic intensity $\vec{H}$ has the components

$$H_r = H_\theta = 0 \qquad (16)$$
$$H_z = H \text{ (constant)}$$

If the body is a perfect conductor of electricity, $\sigma \to \infty$ and the equation (5) gives,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{u}}{\partial t} \times \vec{B} = \left[-\frac{\mu_e}{c} H \frac{\partial v}{\partial t}, 0, 0 \right] \qquad (17)$$

Eliminating $\vec{E}$ from (2) and (17) and using (4) we get,

$$\vec{h} = \left[0, H \frac{\partial v}{\partial z}, 0 \right] \qquad (18)$$

The equations (1) & (4) together with the equation (18) give,

$$\vec{J} \times \vec{B} = \left[0, -\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2}, 0 \right] \qquad (19)$$

Using equations (7), (8), (10), (15) and (6), we get from equation (9),

$$2\mu\left(1+m_2\frac{\partial}{\partial t}\right)\left\{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right\} + 2\left(1+m_2\frac{\partial}{\partial t}\right)$$
$$\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\frac{\partial \mu}{\partial r} - \left(1+m_1\frac{\partial}{\partial t}\right)\left(\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} + \rho\frac{\partial^2 v}{\partial t^2}\right) = 0 \qquad (20)$$

Therefore,

$$v = (A \cos qz + B \sin qz)\, F(r)\, e^{ipt} \qquad (21)$$

satisfies the equation (20) provided $F(r)$ satisfies the equation,

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left\{ \frac{(1+m_1 ip)}{2\mu(1+m_2 ip)}\frac{H^2q^2}{4\pi} - q^2 + \frac{(1+m_1 ip)}{2\mu(1+m_2 ip)}\rho p^2 - \frac{1}{r^2} \right\} F(r) +$$
$$+\left(\frac{dF}{dr} - \frac{F}{r}\right)\frac{1}{\mu}\frac{d\mu}{dr} = 0 \qquad (22)$$

But the surfaces $z = \pm h$ are free from stresses. Thus,

$$(\sigma_{\theta z})_{z = \pm h} = 0 \qquad (23)$$

Hence, we have either,

$B = 0$ and $q = \frac{n\pi}{h}$, ($n=0$ or an integer. This is the symmetric mode of vibration.)

or,

$A = 0$ and $q = (2n+1)\frac{\pi}{2h}$, ($n = 0$ or an integer. This is antisymmetric mode of vibration.)

In view of the equations (17) and (18) we take

$$\vec{E}^* = [E^*, 0, 0]$$
$$\vec{h}^* = [0, h^*, 0]$$

Hence the equations (11) to (14) take the following form,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E^* = 0 \tag{24}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) h^* = 0 \tag{25}$$

and

$$\frac{\partial h^*}{\partial t} = - \frac{c}{r} \frac{\partial E^*}{\partial z} \tag{26}$$

For free torsional vibrations we seek the solutions of (26) in the form,

$$h^* = (A \cos qz + B \sin qz)\, h_0^*(r)\, e^{ipt} \tag{27}$$
$$E^* = (A \cos qz + B \sin qz)\, E_0^*(r)\, e^{ipt}$$

Thus the equation (26) together with (27) give,

$$\frac{d^2 h_0^*}{dr^2} + \frac{1}{r} \frac{dh_0^*}{dr} + \frac{p^2}{c^2} h_0^* = 0 \tag{28}$$

and

$$\frac{d^2 E_0^*}{dr^2} + \frac{1}{r} \frac{dE_0^*}{dr} + \frac{p^2}{c^2} E_0^* = 0 \tag{29}$$

The boundary condition for the slab with cylindrical hole is given by

$$\sigma_{r\theta} - T_{r\theta} - \overset{*}{T}_{r\theta} = 0 \text{ on } r = a \tag{30}$$

where $T_{r\theta}$, $\overset{*}{T}_{r\theta}$ are Maxwell's stress tensors in the body and in vacuum. We can easily verify that,

$$T_{r\theta} = \overset{*}{T}_{r\theta} = 0 \tag{31}$$

and hence the boundary conditions (30) reduces to

$$\sigma_{r\theta} = 0 \text{ on } r = a \tag{32}$$

METHOD OF SOLUTION

Case I : In this case we assume

$$\left.\begin{aligned} \mu &= \mu_0 \left(\frac{r}{a}\right)^2 \\ \rho &= \rho_0 \left(\frac{r}{a}\right)^2 \end{aligned}\right\} \tag{33}$$

where μ_0, ρ_0 are constants and r is the radius vector. The equation (22) with the help of (33) takes the form,

$$\frac{d^2 F}{dr^2} + \frac{3}{r} \frac{dF}{dr} + \left(m^2 - \frac{\lambda_1^2}{r^2} \right) F(r) = 0 \tag{34}$$

where

$$m^2 = \frac{(1 + m_1 ip)\,\rho_0 p^2}{2\mu_0\,(1 + m_2 ip)} - q^2 \tag{35}$$

$$\lambda_1^2 = 3 - \frac{(1 + m_1 ip)}{2\mu_0\,(1 + m_2 ip)}\;\frac{H^2 q^2 a^2}{4\pi}$$

putting

$$F(r) = \frac{1}{r}\,\psi(r) \tag{36}$$

in the equation (34), we obtain,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \left(m^2 - \frac{\lambda^2}{r^2}\right)\psi(r) = 0 \tag{37}$$

where,

$$\lambda^2 = 1 + \lambda_1^2 \tag{38}$$

Since v is finite on the axis of the cylinder, it necessitates that the solution of the equation (37) must be of the form,

$$\psi(r) = J_\lambda(mr) \tag{39}$$

where J_λ is the Bessel function of order λ. Substituting $\psi(r)$ in the equation (36) we obtain,

$$F(r) = \frac{1}{r}\,J_\lambda(mr) \tag{40}$$

The solution therefore is of either of the following forms :

$$v = A \cos qz \,.\, \frac{1}{r} J_\lambda(mr)\, e^{ipt} \quad \text{with } q = \frac{n\pi}{2h} \tag{41}$$

$$v = B \sin qz \,.\, \frac{1}{r} J_\lambda(mr)\, e^{ipt} \quad \text{with } q = \frac{(2n+1)\,\pi}{2h} \tag{42}$$

The frequency equation is obtained from the stress free condition on the hole $r = a$. Thus,

$$(\sigma_{r\theta})_{r=a} = 0$$

leads to the equation

$$ma\, J_{\lambda+1}(ma) = (2-\lambda)\, J_\lambda(ma) \tag{43}$$

The lowest symmetric mode with $n = 0$ has the solution

$$v = B\, r^{-1} J_\lambda\left\{\left(\frac{(1+m_1 ip)\,\rho_0 p^2}{2\mu_0(1+m_2 ip)}\right)^{\frac{1}{2}} r\right\} e^{ipt} \tag{44}$$

with the frequency equation,

$$\sqrt{\frac{(1+m_1 ip)\,\rho_0 p^2}{2\mu_0(1+m_2 ip)}}\,.\,a\,J_{\lambda+1}\left(\sqrt{\frac{(1+m_1 ip)\,\rho_0 p^2}{2\mu_0(1+m_2 ip)}}\,.\,a\right) = (2-\lambda)\,J_\lambda\left(\sqrt{\frac{(1+m_1 ip)\,\rho_0 p^2}{2\mu_0(1+m_2 ip)}}\,.\,a\right) \tag{45}$$

Case II : In this case we suppose that the density of the slab is constant and the shear modulus varies as

$$\mu = \mu_0 \left(\frac{r}{a} \right)^2 \tag{46}$$

The equation (22) together with (46) reduces to

$$\frac{d^2 F}{dr^2} + \frac{3}{r} \frac{dF}{dr} + \left(l^2 - \frac{\beta^2}{r^2} \right) F(r) = 0 \tag{47}$$

where,

$$\beta^2 = 3 + \left(\rho p^2 a^2 - \frac{H^2 q^2 a^2}{4\pi} \right) \left(\frac{1 + m_1 ip}{2\mu_0 (1 + m_2 ip)} \right) \tag{48}$$

$$l^2 = - q^2$$

Proceeding exactly as in the previous case we find the solution of the equation (47) as

$$F(r) = \frac{1}{r} J_\mu (lr) \tag{49}$$

where,

$$\mu = 1 + \beta^2 \tag{50}$$

The frequency equation in this case is obtained, from the condition,

$$(\sigma_{r\theta})_{r = a} = 0$$

as

$$la \, J_{\mu + 1} (la) = (2 - \mu) J_\mu (la) \tag{51}$$

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