A NOTE ON WAVES GENERATED AT A LIQUID-SOLID INTERFACE

K. S. HARINATH

Regional Engineering College, Tiruchirapalli

(Received 13 August 1975)

In this note, a liquid layer above a thermoelastic halfspace is considered. The frequency equation for the waves generated at the interface is obtained and a few limiting cases are discussed.

In a previous paper[1] the author had studied the frequency equation for waves generated at the interface between two halfspaces one assumed elastic and the other thermoelastic. In this note, it is proposed to study the surface waves propagated at the interface between a liquid layer and a thermoelastic halfspace. Various limiting cases are discussed[1]. For the classical case of a liquid layer above an elastic halfspace see the lucid presentation[2] and also the discussion which follows, on geophysical studies of water-covered areas.

BASIC EQUATIONS

We set up a cartesian coordinate system (x,y,z) in such a manner that the thermoelastic halfspace occupies the region $z \geqslant H$, the liquid layer occupies $0 \leqslant z < H$ and $z = 0$ is the free surface. So, the thickness of the liquid layer is H and $z = H$ is the interface. For the surface waves propagated in the x-direction the displacement components are given by;

$$U = \phi_x - \psi_z, \qquad V = 0, \qquad W = \phi_z + \psi_x \tag{1}$$

where the potential functions ϕ, ψ and the temperature perturbation T satisfy the equations

$$\left.\begin{aligned} &\rho\alpha^2 (\phi_{xx} + \phi_{zz}) - \gamma T - \rho\ddot{\phi} = 0 \\ &\beta^2 \ (\psi_{xx} + \psi_{zz}) - \ddot{\psi} = 0 \\ &k (T_{xx} + T_{zz}) - \rho s \dot{T} - \gamma\, T_0 \,(\dot{\phi}_{xx} + \dot{\phi}_{zz}) = 0 \end{aligned}\right\} \tag{2}$$

where ρ is the density of the thermoelastic medium, α is the isothermal (compressional) wave velocity, β is the shear wave velocity, k is the coefficient of thermal conductivity, s is the specific heat at constant strain, γ is the ratio of the coefficient of volume expansion to isothermal compressibility and T_0 is the equilibrium temperature prior to the appearance of a disturbance throughout the region $z \geqslant 0$.

Assuming the solutions of equations (2) in the form

$$(T, \phi, \psi) = (T', \phi', \psi') \exp [i\,(\delta x - \omega t)] \tag{3}$$

we obtain the solutions given by equations (4) below, where it is clear that $T, \phi, \psi \to 0$ as $z \to \infty$ in order that surface waves may be described.

$$T = \frac{\rho}{\gamma} \left[A e^{-\lambda_1 z} \ (\omega^2 - \alpha^2 q_1^2) + B e^{-\lambda_2 z} \ (\omega^2 - \alpha^2 q_2^2) \right] \ e^{i(\delta x - \omega t)}$$

$$\phi = \left[\quad Ae^{-\lambda_1 z} + Be^{-\lambda_2 z} \right] e^{i(\delta x - \omega t)}$$

$$\psi = C \, . \, e^{-\lambda_3 z} \, . \, e^{i(\delta x - \omega t)} \tag{4}$$

where $\lambda_j^2 = \delta^2 - q_j^2$, Re $(\lambda_j) \geqslant 0$, $j = 1, 2, 3$ and q_1^2, q_2^2 are the roots of the equation

$$k\alpha^2 q^4 - q^2 [k\omega^2 + i\omega\rho s\alpha^2 (1 + \epsilon)] + i\omega^3 \rho s = 0 \tag{5}$$

where $\epsilon = \gamma^2 T_0 / \rho^2 s \alpha^2$ is the coupling constant and $q_3^2 = \omega^2/\beta^2$.

The components of displacement in the liquid layer of thickness H is given by

$$U' = \phi'_x, \qquad V' = 0, \qquad W' = \phi'_z \tag{6}$$

where the potential function ϕ' satisfies

$$\mu^2 (\phi'_{xx} + \phi'_{zz}) = \phi' \tag{7}$$

where μ is the velocity of the waves in the liquid.]

Let ρ' be the density of the liquid, p its hydrostatic pressure and $m^2 = \delta^2 - \omega^2/\mu^2$ with Re (m) non-negative. The liquid layer is assumed to be maintained at the constant temperature T_0.

Assuming again a simple harmonic time dependence factor exp $(-i\omega t)$, we obtain

$$\phi' = [A'e^{-mz} + B'e^{mz}] \, e^{i(\delta x - \omega t)}$$

$$p = i\omega\rho'\phi' \tag{8}$$

The constants A, B, C in (4) and A', B' in (8) have to be determined by using the boundary conditions.

BOU DARY CONDITIONS

We impose the following conditions :

(a) At the free surface $p = 0$

(b) Across the interface $z = H$

$\sigma_{zz} = p, \quad \sigma_{xz} = o, \quad T = o \quad W = W',$

where the stresses σ_{zz}, + σ_{xz} are given by :

$$\sigma_{zz} = \rho\alpha^2(\phi_{xx} + \phi_{zz}) \; 2\rho\beta^2(\psi_{xz} - \phi_{xx}) - \gamma T$$

$$\sigma_{xz} = \rho\beta^2(2\phi_{xz} + \psi_{xx} - \psi_{zz}) \tag{9}$$

It is to be noted that no condition on the tangential displacements can be imposed at the interface.

FREQUENCY EQUATION

By the use of the above five boundary conditions on $z = 0$ and $z = H$ we obtain homogeneous equations for the five constants A, B, C, A' B'. These have a nontrivial solution only when the matrix of the coefficients of A, B, C, A', B' is singular. This determines the frequency equation to be

$$\begin{vmatrix} e^{mH} & e^{-mH} & 0 & 0 & 0 \\ i\omega\rho' & i\omega\rho' & \rho(2\beta^2\delta^2-\omega^2) & \rho(2\beta^2\delta^2-\omega^2) & -2i\lambda_3\delta\rho\beta^2 \\ 0 & 0 & 2i\delta\lambda_1\beta^2 & 2i\delta\lambda_2\beta^2 & 2\beta^2\delta^2-\omega^2 \\ 0 & 0 & \omega^2-\alpha^2q_1^2 & \omega^2-\alpha^2q_2^2 & 0 \\ m & -m & -\lambda_1 & -\lambda_2 & i\delta \end{vmatrix} = 0$$

OR

$$\left[\rho'\omega^3 \tan(imH) - 4\rho m\delta^2\lambda_3\beta^4\right]\left[\alpha^2(\lambda_1q_2^2 - \lambda_2q_1^2) - \omega^2(\lambda_1 - \lambda_2)\right] =$$

$$= \rho\alpha^2 m\,(2\beta^2\delta^2 - \omega^2)^2\,(q_1^2 - q_2^2) \tag{10}$$

It is to be noted that equation (10) is the thermœlastic version of the equation (4 –154), given by Ewing[2], *et.al.* As a remark, we can add that (10) cannot be deduced from the more general frequency equation for waves at the interface of a solid halfspace with a solid layer above by merely assuming one of the Lame constants to be zero. Hence, a separate study of the liquid-solid interface is necessary.

LIMITING CASES

There are various limiting cases and most of these are discussed[1]. Here we merely discuss the forms of the frequency equation for large and small frequencies (ω) when the wave number (δ) is fixed. The other cases for large and small wave lengths when the frequency is fixed and also for small values of the coupling constant may be obtained on lines similar to those in the author's earlier paper[1].

(*i*) *For large frequencies*: Suppose $\omega >> 1$ and δ is fixed. In case the liquid layer is also very thin (*i.e.* $H << 1$), the frequency equation (10) takes the form

$$2ik\delta\,(\rho'H - \rho\alpha).c^3 + \rho\rho^1 s\epsilon H\alpha^2.c^2 - 8i\rho k\delta\beta^3.c - 4\rho^2 s\alpha^2\epsilon\beta^3 = 0 \tag{11}$$

where $c = \omega/\delta$ measures the phase-velocity ($= [\mathrm{Re}\,(1/c)]^{-1}$) and the attenuation in the x-direction ($= \omega\,[\mathrm{I_m}\,(1/c)]$).

Putting $\epsilon = 0$ in equation (11) we obtain

$$c^2\,[\rho'H - \rho\alpha] = 4\rho\beta^3 \tag{12}$$

which is nothing but equation (4—154) in Ewing[2], *et. al.* for $\omega >> 1$ and $H << 1$. In the general case when $\omega >> 1$, the frequency equation (10) takes the form :

$$[\rho'\omega\mu\tan(\omega H/\mu) - 4\rho\delta^2\beta^3]\,(\rho s\epsilon\alpha^2 + 2ik\omega) - 2i\rho k\alpha\omega^3 = 0 \tag{13}$$

Various cases of interest, *viz.* $0 \leqslant H \leqslant \left|\dfrac{\pi\mu}{2\omega}\right|$ for real values of ω of (13) may be discussed.

(*ii*) *For an incompressible solid*: For an incompressible solid we know that the compressional wave velocity is infinite. Hence if the thermœlastic halfspace $z \leqslant H$ is incompressible equation (10) takes the form

$$\rho m\,(2\beta^2\delta^2 - \omega^2)^2 + \rho'\delta\omega^3\tan(imH) - 4\rho m\delta^3\beta^4\sqrt{\delta^2 - \omega^2/\beta^2} = 0 \tag{14}$$

For large or small values of the frequency ω when δ is fixed (14) may be evaluated.

(*iii*) *For small frequencies*: Suppose $\omega << 1$ and δ is fixed. Then the frequency equation (10) takes the form

$$\rho\,(2\beta^2\delta^2 - \omega^2)^2 + \rho'\omega^3 \tan(i\delta H) - 4\rho\delta^4\beta^4 = 0 \tag{15}$$

which shows that the phase velocity and the attenuation are independent of the compressional wave velocity α and hence for small values of the frequencies, (14) reduces to (15) simply by replacing m by δ and neglecting ω^2/β^2.

CONCLUSIONS

In all the limiting cases considered above it can be observed that the wave velocity depends on the frequency and hence there is dispersion. In other words in equations (11), (13), (14), (15) ω and δ are always coupled whereas in the elastic case (equation (12)) ω is proportional to δ. Thus the dispersion which is absent in equation (12) but present in the other equations is due to the introduction of temperature perturbation. Further for small frequencies, the compressibility of the solid has no influence on the frequency equation as shown by equations (14) and (15). The above results have direct bearing on defence studies of underground explosions, whether they are caused by nuclear devices or conventional ones. These enable us to compute the intensity and the frequency of the shock waves thus generated.

ACKNOWLEDGEMENT

The author is grateful to Prof. Dr. V. R. Thiruvenkatachar for many helpful discussions.

REFERENCES

1. HARINATH, K. S., *Def. Sci. J.*, 24 (1975), 67.
2. EWING, W.M.D. JARDETZKY, W.S. & PRESS, F., 'Elastic waves in layered media'. (McGraw Hill, New York), 1957, p. 611.
3. CHADWICK, P., *Prog. Solid Mech.*, 1, (1960) 265.