A NOTE ON THE MODE OF VIBRATION IN LAYERED FLAT-TOPPED CONICAL HILL (DECCAN TRAP TYPE) AND THE STRESSES GENERATED IN EACH LAYER—PART I

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In this paper the authors have calculated the displacements and stresses at each layer of a n-layered flat-topped conical hill while the stresses of the type of a confined explosions are applied at the bottom of the hill. The solution is obtained in terms of Bessel's function and the condition of fracturing is also specified.

Deccan traps of basaltic composition covers about 200,000 sq. miles of the peninsular India and conceal a large amount of precious mineral resources. The traps occur as sheets or layers of different thicknesses. A traverse through the country would reveal to any observer the layered nature of the hills which in general have flat tops and often assumes conical shapes.

As a preliminary investigation on the nature of deformation etc. impressed on the Deccan traps by different geodynamic or artificial elastic stresses, we take up the case of the isolated hill with layered flows, as commonly found in Gujrat, Maharashtra, Madhya Pradesh etc.

In the present study we studied the nature of displacement and stresses at different homogeneous layers, when the stress is applied in the form of a confined explosion just at the bottom of the Deccan trap. The basalts being fine-grained and compact are considered to be elastically isotropic.

In this paper, we take up a n-layered case, where n is observed in Deccan region, When n is very large, we assume that the displacement is negligible and is equal to zero. We, therefore, in the present model investigated the effect upto $(n-1)$th level.

Fig. 1 represents the case as postulated and different layers are marked as 1, 2, 3,..........n. The stress in the form of radial vibration is assumed to be generated at the centre of the base. That is the stress $(\widehat{rr})_1$ is applied at $r=a_1$. Since the vibration studied is radial, the co-ordinates we take are spherical polar co-ordinates (r, θ, ϕ). The vertical axis of the cone passing through the explosion centre is the vertical axis of the system.

The modulus of rigidity, μ_N, and density, ρ_n, of N-th layer ($N=1, 2, 3,$................n) are constants.

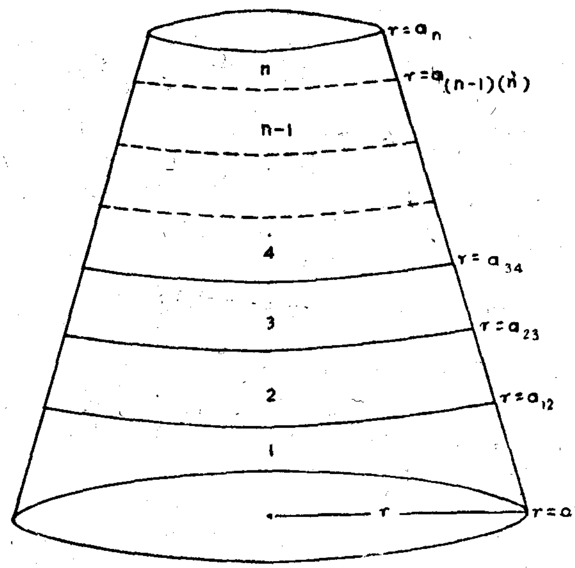

Fig. 1—Flat topped conical hill with n horizontal layers numbered 1,2,3,...,n and γ=aij, for i, j=1,2,3,..., n denotes the distance of surface of separation of two consecutive layers i and j from the vertex of conical hill.

FUNDAMENTAL EQUATIONS

Stress equations of motions[1] are :

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \widehat{r\phi}}{\partial \phi} + \frac{1}{r}\left(2\widehat{rr} - \widehat{\theta\theta} - \widehat{\phi\phi} + \widehat{r\theta}\cot\theta\right) = \rho\frac{\partial^2 u_r}{\partial t^2} \tag{1}$$

$$\frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \widehat{\theta\phi}}{\partial \phi} + \frac{1}{r}\left\{(\widehat{\theta\theta} - \widehat{\phi\phi})\cot\theta + 3\widehat{r\theta}\right\} = \rho\frac{\partial^2 u_\theta}{\partial t^2} \tag{2}$$

$$\frac{\partial \widehat{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \widehat{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \widehat{\phi\phi}}{\partial \phi} + \frac{1}{r} \left(3 \widehat{r\phi} + 2 \widehat{\theta\phi} \cot \theta \right) = \rho \frac{\partial^2 u_\phi}{\partial t^2} \tag{3}$$

Since we are assuming the radial vibrations, displacement, $(u_\theta)_N = (u_\phi)_N = 0$ and $(u_r)_N = (u)^2 = R_N(r) e^{ipt}$ (where $N=1, 2, 3, \ldots\ldots\ldots n$; R is a function of r only).

Hence the stress components

$$(\widehat{r\theta})_N = (\widehat{\theta\phi})_N = (\widehat{r\phi})_N = 0$$

and

$$\left.\begin{aligned} (\widehat{rr})_N &= \mu_N \frac{\partial u_N}{\partial r} = \mu_N \frac{dR_N}{dr} e^{ipt} \\ (\widehat{\theta\theta})_N &= \mu_N \frac{u_N}{r} = \mu_N \frac{R_N}{r} e^{ipt} \\ (\widehat{\phi\phi})_N &= \mu_N \frac{u_N}{r} = \mu_N \frac{R_N}{r} e^{ipt} \end{aligned}\right\} \tag{4}$$

Since the vibration is radial, equations (2) and (3) are identically zero and the remaining one reduces to the form :

$$\frac{\partial (\widehat{rr})_N}{\partial r} + \frac{1}{r} \left\{ 2(\widehat{rr})_N - (\widehat{\theta\theta})_N - (\widehat{\phi\phi})_N \right\} = \rho_N \frac{\partial^2 u_N}{\partial t^2}$$

or

$$\frac{d^2 R_N}{dr^2} + \frac{2}{r} \frac{dR_N}{dr} + \left(\nu^2_N - \frac{2}{r^2} \right) R_N = 0$$

Therefore

$$R_N = r^{-1/2} \left\{ A_N J_{3/2} (\nu_N r) + B_N Y_{3/2} (\nu_N r) \right\}$$

Hence

$$(n)_N = r^{-1/2} \left\{ A_N J_{3/2} (\nu_N r) + B_N Y_{3/2} (\nu_N r) \right\} e^{ipt} \tag{5}$$

(for $N=1, 2, 3, \ldots\ldots\ldots, n$)

where

$$\nu_N = \sqrt{\frac{\rho_N p^2}{\mu_N}}$$

Boundary conditions

(I) $\quad u_N = 0 \quad \text{on} \quad r = a_n$

(II) $\quad (\widehat{rr})_1 = S\, e^{ipt} \quad \text{on} \quad r = a_1$

where S is a constant

(III) (a) $(\widehat{rr})_N = (\widehat{rr})_{N+1}$ (b) $u_N = u_{N+1}$ on $r = a_{N(N+1)}$ for $N = 1, 2, 3, \ldots\ldots\ldots (n-1)$.

Applying boundary condition (I)

$$A_n = - \frac{B_n Y_{3/2} (\nu_n a_n)}{J_{3/2} (\nu_n a_n)} \tag{6}$$

Applying boundary condition (II)

$$A_1 \left\{ J_{3/2} (\nu_1 a_1) - \nu_1 a_1 J_{5/2} (\nu_1 a_1) \right\} + B_1 \left\{ Y_{3/2} (\nu_1 a_1) - \nu_1 a_1 Y_{5/2} (\nu_1 a_1) \right\} = = \frac{S a_1^{3/2}}{\mu_1} \tag{7}$$

Applying boundary condition III(a)

$$A_N P_N^{(N+1)} J_{3/2}\left(\nu_n a_n \right) + B_N Q_N^{(N+1)} J_{3/2}\left(\nu_n a_n \right) - \frac{\mu_{N+1}}{\mu_N} B_n \cdot \left\{ Q_{N+1}^{(N)} J_{3/2}\left(\nu_n a_n \right) - P_{N+1}^{(N)} Y_{3/2}\left(\nu_n a_n \right) \right\} = 0 \tag{8}$$

where

$$P_N^{(N+1)} = J_{3/2}\left(\nu_N a_{N(N+1)} \right) - \nu_N a_{N(N+1)} J_{5/2}\left(\nu_N a_{N(N+1)} \right)$$

$$Q_N^{(N+1)} = Y_{3/2}\left(\nu_N a_{N(N+1)} \right) - \nu_N a_{N(N+1)} Y_{5/2}\left(\nu_N a_{N(N+1)} \right)$$

$$P_{N+1}^{(N)} = J_{3/2}\left(\nu_{N+1} a_{N(N+1)} \right) - \nu_{N+1} a_{N(N+1)} J_{5/2}\left(\nu_{N+1} a_{N(N+1)} \right)$$

$$Q_{N+1}^{(N)} = Y_{3/2}\left(\nu_{N+1} a_{N(N+1)} \right) - \nu_{N+1} a_{N(N+1)} Y_{3/2}\left(\nu_{N+1} a_{N(N+1)} \right)$$

for $N = 1, 2, 3, \ldots\ldots, (n-1)$.

Applying boundary condition (III) (b) we get

$$A_N J_{3/2}\left\{ \nu_N a_{N(N+1)} \right\} J_{3/2}\left(\nu_n a_n \right) + B_N Y_{3/2}\left\{ \nu_N a_{N(N+1)} \right\} J_{3/2}\left(\nu_n a_n \right) - - B_N \left[Y_{3/2}\left\{ \nu_{N+1} a_{N(N+1)} \right\} J_{3/2}\left(\nu_n a_n \right) - J_{3/2}\left\{ \nu_{N+1} a_{N(N+1)} \right\} Y_{3/2}\left(\nu_n a_n \right) \right] = 0 \tag{9}$$

If we eliminate the constants from equations (8) & (9) after putting $N=n-1$ and using equation (6), we get

$$A_{n-1} = \frac{B_n}{J_{3/2}\left(\nu_n a_n \right)} \cdot \frac{D_{A_{n-1}}}{D_{B_n}^{(n)}} \tag{10}$$

$$B_{n-1} = \frac{B_n}{J_{3/2}\left(\nu_n a_n \right)} \frac{D_{B_{n-1}}}{D_{B_n}^{(n)}} \tag{11}$$

$$D_{A_{n-1}} = \frac{\mu_n}{\mu_{n-1}} Y_{3/2}\left(\nu_{n-1} a_{(n-1)n} \right) \left[Q_n^{(n-1)} J_{3/2}\left(\nu_n a_n \right) - P_n^{(n-1)} Y_{3/2}\left(\nu_n a_n \right) \right] - - Q_{n-1}^{(n)} \left[Y_{3/2}\left(\nu_n a_{(n-1)n} \right) J_{3/2}\left(\nu_n a_n \right) - J_{3/2}\left(\nu_n a_{(n-1)n} \right) Y_{3/2}\left(\nu_n a_n \right) \right]$$

$$D_{B_{n-1}} = P_{n-1}^{(n)} \left[Y_{3/2}\left(\nu_n a_{(n-1)n} \right) J_{3/2}\left(\nu_n a_n \right) - J_{3/2}\left(\nu_n a_{(n-1)n} \right) Y_{3/2}\left(\nu_n a_n \right) \right] -$$

$$- \frac{\mu_n}{\mu_{n-1}} J_{3/2}\left(\nu_{N-1} \; a_{(n-1)n} \right)\left[Q_n^{(n-1)} \; J_{3/2}\left(\nu_n \; a_n \right) - P_n^{(n-1)} \; Y_{3/2}\left(\nu_n \; a_n \right)\right]$$

$$D_{B_n}^{(n)} = P_{n-1}^{(n)} \; Y_{3/2}\left(\nu_{n-1} \; a_{(n-1)n}\right) - Q_{n-1}^{(n)} \; J_{3/2}\left(\nu_{n-1} \; a_{(n-1)n}\right).$$

Substituting $N = n-2$ and using (9) and (10) from equations (7) & (8), we obtain

$$A_{n-2} = \frac{B_n}{J_{3/2}\left(\nu_n \; a_n \right) D_{B_n}^{n}} \quad \frac{D_{A_{n-2}}}{D_{B_n}^{(n-1)}} \tag{12}$$

$$B_{n-2} = \frac{B_n}{J_{3/2}\left(\nu_n \; a_n \right) D_{B_n}^{n} \; D_{B_n}^{(n-1)}} \; D_{B_{n-2}} \tag{13}$$

where

$$D_{A_{n-2}} = \frac{\mu_{n-1}}{\mu_{n-2}} Y_{3/2}\left(\nu_{n-2} \; a_{(n-2)(n-1)}\right) \left[P_{n-1}^{(n-2)} D_{A_{n-1}} + \right.$$
$$\left. + Q_{n-1}^{(n-2)} \; D_{B_{n-1}} \right] - Q_{n-2}^{(n-1)}\left[J_{3/2}\left(\nu_{n-1} \; a_{(n-2)(n-1)}\right) D_{A_{(n-1)}} + \right.$$
$$\left. + Y_{3/2}\left(\nu_{n-1} \; a_{(n-2)(n-1)} \right) D_{B_{(n-1)}} \right]$$

$$D_{B_{n-2}} = P_{n-2}^{(n-1)}\left[J_{3/2}\left\{ \nu_{n-1} \; a_{(n-2)(n-1)}\right\} D_{A_{n-1}} + \right.$$
$$\left. + Y_{3/2}\left\{ \nu_{n-1} \; a_{(n-2)(n-1)}\right\} D_{B_{n-1}} \right] - \frac{\mu_{n-1}}{\mu_{n-2}} \; J_{3/2}\left\{ \nu_{n-2} \; a_{(n-2)(n-1)}\right\} \cdot$$
$$\cdot \left[P_{n-1}^{(n-2)} \; D_{A_{n-1}} + Q_{n-1}^{(n-2)} \; D_{B_{n-1}} \right]$$

$$D_{B_n}^{(n-1)} = \left[P_{n-2}^{(n-1)} \; Y_{3/2}\left\{ \nu_{n-2} \; a_{(n-2)(n-1)}\right\} - \right.$$
$$\left. - Q_{n-2}^{(n-1)} \; J_{3/2}\left\{ \nu_{n-2} \; a_{(n-2)(n-1)}\right\}\right]$$

Proceedings in this manner, we get

$$A_2 = \frac{B_n \; D_{A_2}}{J_{3/2}\left(\nu_n \; a_n \right) D_{B_n}^{(n)} \; D_{B_n}^{(n-1)} \; D_{B_n}^{(n-2)} \; \ldots \; D_{B_n}^{(3)}} \tag{14}$$

$$B_2 = \frac{B_n \; D_{B_2}}{J_{3/2}\left(\nu_n \; a_n \right) D_{B_n}^{(n)} \; D_{B_n}^{(n-1)} \; \ldots \ldots \; D_{B_n}^{(3)}} \tag{15}$$

where

$$D_{A_2} = \frac{\mu_3}{\mu_2} Y_{3/2}\left(\nu_2 a_{23} \right) \left[P_3^{(2)} D_{A_3} + Q_3^{(2)} D_{B_3} \right] +$$
$$+ Q_2^{(3)} \left[J_{3/2}\left(\nu_3 a_{23} \right) D_{A_3} + Y_{3/2}\left(\nu_3 a_{23} \right) D_{B_3} \right]$$

$$D_{B_2} = P_2^{(3)} \left[J_{3/2}\left(\nu_3 a_{23} \right) D_{A_3} + Y_{3/2}\left(\nu_3 a_{23} \right) D_{B_3} \right] -$$
$$- \frac{\mu_3}{\mu_2} J_{3/2}\left(\nu_2 a_{23} \right) \left[P_3^{(2)} D_{A_3} + Q_3^{(2)} D_{B_3} \right]$$

Substituting $N=1$ and using equations (14), (15) and (6), we get,

$$B_n = \frac{S a_1^{3/2}}{\mu_1 E} J_{3/2}\left(\nu_n a_n \right) D_{B_n}^{(n)} D_{B_n}^{(n-1)} \cdots\cdots D_{B_n}^{(3)} D_{B_n}^{(2)}$$

$$A_1 = \frac{S a_1^{3/2}}{\mu_1 E} D_{A_1}$$

$$B_1 = \frac{S a_1^{3/2}}{\mu_1 E} D_{B_1}$$

where

$$E = \left[J_{3/2} (\nu_2 a_{12}) D_{A_2} + Y_{3/2} (\nu_2 a_{12}) D_{B_2} \right]\left[P_1^{(2)} Q_1 - Q_1^{(2)} P_1 \right] -$$
$$- \frac{\mu_2}{\mu_1} \left[P_2^{(1)} D_{A_2} + Q_2^{(1)} D_{B_2} \right] \left[Q_1 J_{3/2} (\nu_1 a_{12}) - P_1 Y_{3/2} (\nu_1 a_{12}) \right]$$

$$D_{A_1} = \frac{\mu_2}{\mu_1} Y_{3/2} \left(\nu_1 a_{12} \right) \left[D_{A_2} P_1^{(2)} + D_{B_2} Q_2^{(1)} \right] - Q_1^{(2)} \left[J_{3/2} \left(\nu_2 a_{12} \right) D_{A_2} \right.$$
$$\left. + Y_{3/2} \left(\nu_2 a_{12} \right) D_{B_2} \right]$$

$$D_{B_1} = P_1^{(2)} \left[D_{A_2} J_{3/2} \left(\nu_2 a_{12} \right) + D_{B_2} Y_{3/2} \left(\nu_2 a_{12} \right) \right] - \frac{\mu_2}{\mu_1} J_{3/2} \left(\nu_1 a_{12} \right)$$
$$\left[D_{A_2} P_2^{(1)} + D_{B_2} Q_2^{(1)} \right]$$

where, again

$$P_1 = J_{3/2} (\nu_1 a_1) - \nu_1 a_1 J_{5/2} (\nu_1 a_1)$$
$$Q_1 = Y_{3/2} (\nu_1 a_1) - \nu_1 a_1 Y_{5/2} (\nu_1 a_1)$$

Thus we can determine the displacement and the stresses at different layers.

The displacements are measured as :

$$u_1 = \frac{S a_1^{3/2}}{\mu_1 E} r^{-\frac{1}{2}} \left[D_{A_1} J_{3/2} (\nu_1 r) + D_{B_1} Y_{3/2} (\nu_1 r) \right] e^{ipt}$$

$$u_2 = \frac{S a_1^{3/2}}{\mu_1 E} D_{B_n}^{(2)} r^{-\frac{1}{2}} \left[D_{A_2} J_{3/2} (\nu_2 r) + D_{B_2} Y_{3/2} (\nu_2 r) \right] e^{ipt}$$

$$u_{n-1} = \frac{S a_1^{3/2}}{\mu_1 E} D_{B_n}^{(2)} D_{B_n}^{(3)} D_{B_n}^{(4)} \cdots\cdots D_{B_n}^{(n-1)} r^{-\frac{1}{2}} \left[D_{A_{n-1}} J_{3/2}\left(\nu_{n-1} r \right) + \right.$$
$$\left. + D_{B_{n-1}} Y_{3/2}\left(\nu_{n-1} r \right) \right] e^{ipt} \qquad (16)$$

The stresses at the boundaries of each layers are :

$$\widehat{(rr)}_{r=a_{12}} = \frac{\mu_2}{\mu_1} D^{(2)}_{B_n} \frac{a_1^{3/2}}{a_{12}^{3/2}} \frac{S}{E} \left[D_{A_2} \left\{ J_{3/2}(\nu_2 a_{12}) - \nu_2 a_{12} J_{5/2}(\nu_2 a_{12}) \right\} + \right.$$
$$\left. + D_{B_2} \left\{ Y_{3/2}(\nu_2 a_{12}) - \nu_2 a_{12} Y_{5/2}(\nu_2 a_{12}) \right\} \right] e^{ipt}$$

$$\widehat{(rr)}_{r=a_{(n-2)(n-1)}} = \frac{\mu_{n-1}}{\mu_1} \frac{a_1^{3/2}}{a^{3/2}_{(n-2)(n-1)}} \frac{S}{E} D^{(2)}_{B_n} D^{(3)}_{B_n} \ldots\ldots D^{(n-1)}_{B_n} .$$

$$\cdot \left[D_{A_{n-1}} \left\{ J_{3/2}\left(\nu_{(n-1)} a_{(n-2)(n-1)} \right) - \nu_{n-1} a_{(n-2)(n-1)} J_{3/2}\left(\nu_{n-1} a_{(n-2)(n-1)} \right) \right\} + \right.$$
$$\left. + D_{B_{n-1}} \left\{ Y_{3/2}\left(\nu_{(n-1)} a_{(n-2)(n-1)} \right) - \nu_{n-1} a_{(n-2)(n-1)} Y_{5/2}\left(\nu_{(n-1)} a_{(n-2)(n-1)} \right) \right\} \right] e^{ipt} \quad (17)$$

DISCUSSION

When the frequency equation (7) does not tally in adjacent layers, rupture would develop. The other mathematical deductions bring out the point that if there are a large number of flow layers so that elastic vibrational energy is dissipated away and becomes negligible at the top surface, it is possible to calculate at different layers from equations (16) and (17) the displacements and the stresses respectively generated by an enclosed explosion at the base of the layers.

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