DUCTILE FRACTURE OF A SPHERICAL SHELL UNDER INTERNAL PRESSURE

V.P. MUTHUSWAMY & P. R. KRISHNAMOORTHY

Regional Engineering College, Tiruchirapalli

(*Received* 15 *February* 1975; *revised* 5 *January* 1976)

Due to monotonically increasing internal pressure, a thick spherical shell of rigid plastic ductile material undergoes plastic deformation. Using Thomas' fracture theory, the critical pressure that causes fracture and the velocity as well as the stress fields during plastic flow, are estimated by analytical method.

In this paper, we consider a spherical shell of rigid plastic ductile material, which is loaded by monotonically increasing internal pressure. The internal pressure is completely specified by a given law. In the initial stages when the internal pressure is gradually increased, the shell will, in general, remain rigid; but if it is increased further, a stage will arise when the inner surface of the body begins to yield, and gradually the plastic deformation, spreads over the whole body.

If the material is perfectly plastic (i.e., non-strain-hardening) and if the resulting change in geometry is disregarded, then under the assumed monotonically increasing internal pressure, there will be considerable amount of plastic flow before the body finally fractures.

The aim of this paper is to determine the stress field and velocity components during plastic flow as also the critical pressure at which the spherical shell fractures.

The following are the basic assumptions :—

1. The volume of the spherical shell does not undergo any change during plastic deformation and flow so that the incompressibility condition is satisfied throughout.

2. The shape of the shell is assumed to be spherical throughout the plastic flow even at the moment when fracture occurs, though it may not be so in the real situation.

3. When fracture occurs, the following fracture condition is satisfied[1]:

$$[(1+q^2)^{\frac{1}{2}} + q] \cdot s_1 - [(1+q^2)^{\frac{1}{2}} - q] \cdot s_3 \geqslant 2Q \tag{1}$$

The constants q and Q are *fracture moduli* for ductile materials where q is much smaller than Q, when expressed in the usual units. Also the quantities s_1 and s_3 are algebraically the greatest and the least of principal stresses at points of the stress field under consideration. No difficulties will arise from strain hardening in the treatment of the ductile solids, since there is sufficient leeway in the inequality in (1) to permit this effect without changing the values of the moduli q and Q.

FORMULATION OF THE PROBLEM

In the fitness of things, the spherical polar co-ordinates are chosen, with the centre of the sphere as the pole. For free use of tensor notation, the co-ordinates r, θ, ϕ are respectively denoted by x^1, x^2, x^3.

In the usual notation, the incompressivility and equilibrium conditions are respectively

$$g^{ij} \, v_{i,j} = 0 \; (i, j = 1, 2, 3) \tag{2}$$

and

$$g^{jk} \, \sigma_{ij,k} = 0 \; (i, j = 1, 2, 3), \; (k = 1, 2, 3) \tag{3}$$

Following Thomas[2], the equation ofthe plastic flow of the material reduces to

$$\epsilon_{ij} = \psi(r) \, . \, \sigma^{*}_{ij} \; (i, j = 1, 2, 3) \tag{4}$$

where ϵ_{ij} is the rate of strain tensor, σ^*_{ij} is the stress deviator and $\psi(r)$ is a positive function of the material.

The boundary conditions for the problem can be stated as :

$$\sigma_{11} = -p, \text{ when } r = a,$$
$$\sigma_{11} = 0, \quad \text{when } r = b \tag{5}$$

and

The values a and b denote the inner and outer radii of the shell at the instant of fracture, and the quantity p, the critical internal pressure which causes fracture in the shell.

As a consequence of the incompressibility assumption, we have the quality

$$(b^3 - a^3) = (b_0^3 - a_0^3) = k_0^3 \text{ (say)} \tag{6}$$

where a_0 and b_0 stand for internal and external radii respectively of the shell, before plastic deformation.

ANALYSIS

At the outset, due to symmetry of the system, we note that σ_{ij} and v_i are functions of r and time t only and also that $\sigma_{ij} = 0$, if $i \neq j$.

The three equilibrium conditions in (3) reduce to

$$\frac{\partial \sigma_{11}}{\partial r} + \frac{2}{r}\sigma_{11} - \frac{2}{r^3}\sigma_{22} = 0 \tag{7}$$

and

$$\frac{\sigma_{22}}{r^2} = \frac{\sigma_{33}}{r^2 \sin^2\theta} \tag{8}$$

The third one turns out to be an identity.

The six equations of the plastic flow (4) can be expressed in the form :

$$\frac{\partial v_1}{\partial r} = \frac{2\psi}{3}\left(\sigma_{11} - \frac{\sigma_{22}}{r^2}\right) \tag{9}$$

$$r v_1 = \frac{1}{3}\psi\,(-r^2\,\sigma_{11} + \sigma_{22}) \tag{10}$$

$$r \sin^2\theta.\ v_1 + \sin\theta \cos\theta.\ v_2 = \frac{1}{3}\,\psi \sin^2\theta\ (-r^2\ \sigma_{11} + \sigma_{22}) \tag{11}$$

$$\frac{1}{2}\ \frac{\partial v_2}{\partial r} - \frac{1}{r}\ v_2 = \psi\,\sigma^*_{12} = 0,\ (\psi \neq 0) \tag{12}$$

$$-\cot\theta.\ v_3 = \psi\,\sigma^*_{23} = 0,\ (\psi \neq 0) \tag{13}$$

$$\frac{1}{2}\ \frac{\partial v_3}{\partial r} - \frac{1}{r} v_3 = \psi\,\sigma^*_{13} = 0,\ (\psi \neq 0) \tag{14}$$

From (10), (11) and (13) we obtain

$$v_2 = 0 \text{ and } v_3 = 0 \tag{15}$$

Since rv_1 is clearly positive, it follows from the equation (10) that $\frac{\sigma_{22}}{r^2} > \sigma_{11}$.

Thus the maximum and minimum principal stress components are respectively

$$s_1 = \frac{\sigma_{22}}{r^2} \text{ and } s_3 = \overline{\overline{\sigma}}_{11}$$

We rewrite the fracture condition (1) as :

$$s_1 = A + B\, s_3$$

where $$A = \frac{2Q}{(1+q^2)^{\frac{1}{2}} + q} \text{ and } B = \frac{(1+q^2)^{\frac{1}{2}} - q}{(1+q)^{\frac{1}{2}} + q} \tag{16}$$

Here we have taken (1) as an equality, as it does not give rise to any qualitative change in the result. Also we note that $B < 1$.

The fracture condition now becomes

$$\frac{\sigma_{22}}{r^2} = A + B\, \sigma_{11} \tag{17}$$

Eliminating σ_{22} between (7) and (17), we find

$$\frac{\partial \sigma_{11}}{\partial r} + \frac{n}{r}\sigma_{11} = \frac{2A}{r} \tag{18}$$

where $$0 < n = (2 - 2B) < 2 \tag{19}$$

The above equation, on integration, gives

$$r^n\, \sigma_{11} = \frac{Q}{q}\, r^n + f(t) \tag{20}$$

where $f(t)$ is the constant of integration depending on t only and $\dfrac{2A}{n} = \dfrac{A}{(1-B)} = \dfrac{Q}{q}$.

Use of the boundary conditions (5) in (20) gives

$$f(t) = -\frac{Q}{q} b^n, \tag{21}$$

and

$$p = \frac{Q}{q}\left[\left(\frac{b}{a}\right)^n - 1\right]$$

or

$$p = \frac{Q}{q}\left[\frac{b^n}{(b^3 - k_0{}^3)^{n/3}} - 1\right] \tag{22}$$

which gives the critical internal pressure when fracture occurs.

The insertion of (21) in (20) gives

$$\sigma_{11} = \frac{Q}{q}\left[1 - \left(\frac{b}{r}\right)^n\right] \tag{23}$$

Again, from (23) and (17), we get

$$\frac{\sigma_{22}}{r^2} = \frac{Q}{q}\left[1 - B\left(\frac{b}{r}\right)^n\right] \tag{24}$$

and from (8),

$$\frac{\sigma_{33}}{r^2 \sin^2\theta} = \frac{Q}{q}\left[1 - B\left(\frac{b}{r}\right)^n\right] \tag{25}$$

From (23), (24) and (25), we conclude that the plastic stress field is dependent only on r and *independent* of t.

In order to investigate the velocity component v_1, we see, in view of (15), that the incompressibility equations (2), when expressed in the explicit form, reduce to a single equation

$$\frac{\partial v_1}{\partial r} + \frac{2v_1}{r} = 0 \tag{26}$$

which, on integration yields :

$$r^2 v_1 = \phi(t) \tag{27}$$

where $\phi(t)$ is the time dependent constant of integration.

Using (10), (23) and (24) in (27) we obtain

$$\phi(t) = \frac{A}{3}\, \psi(r)\, b^n\, r^{3-n} \tag{28}$$

Since $\phi(t)$ is independent of r, the (28) will be valid only if $\psi(r) = \frac{C_0}{r^{3-n}}$, where C_0 is some constant.

we let $\psi_0 = \psi(b_0) = \frac{C_0}{b_0^{3-n}}$, then

$$\psi(r) = \psi_0 \left(\frac{b_0}{r}\right)^{3-n} \tag{29}$$

The constant ψ_0 depending upon the material is a positive quantity and can be assumed to be known.

Now, (27) together with (28) and (29) gives the velocity component

$$v_1 = \frac{A\, \psi_0\, b_0^{3-n}\, b^n}{3} \cdot \frac{1}{r^2} \tag{30}$$

In all the equations (22) through (25) and (30), the unknown quantity b, namely the outer radius of the shell at the instant of fracture, is involved. The work will be complete if the value of b is expressed in terms of the known quantities ψ_0; Q, q, a_0, b_0 etc. To this end we now make use of the energy principle[3].

The internal pressure P is a function $P(t)$ of time t. But the parameter t can very well be replaced by any other parameter which is monotonically increasing as t increases. The inner radius ρ at any time t suits well for this purpose. Accordingly, we denote the internal pressure function as $P(\rho)$ and specify it by some simple law, say

$$P(\rho) = \frac{P_0}{a_0}\, \rho \tag{31}$$

Initially, when $\rho = a_0$, $P(a_0) = P_0 =$ constant.

The value of P_0 (vide Hoffman & Sachs)[4] is found[4] to be $4K \log\left(\frac{b_0}{a_0}\right)$.

Now the total energy input E is given by

$$E = \int_{a_0}^{a} P(\rho)\, 4\pi\rho^2\, d\rho = \int_{a_0}^{a} 4\pi\, \frac{P_0}{a_0}\, \rho^3\, d\rho$$

Therefore, time rate of total energy input

$$\frac{dE}{d\rho} = \int_{a_0}^{a} 4\pi \frac{P_0}{a_0}\left\{\frac{\partial}{\partial \rho}(\rho^3)\right\} d\rho = \frac{4\pi P_0}{a_0}(a^3 - a_0^3)$$

Also the total plastic work done over the whole volume in unit time, denoted by W^p, is calculated as :

$$W^p = \int_a^b \left(\sigma_{11}\epsilon_{11} + \frac{\sigma_{22}}{r^2}\frac{\epsilon_{22}}{r^2} + \frac{\sigma_{33}}{r^2\sin^2\theta}\frac{\epsilon_{33}}{r^2\sin^2\theta}\right) 4\pi r^2\, dr$$

$$= \int_a^b \left(\sigma_{11}\frac{\partial v_1}{\partial r} + 2\frac{\sigma_{22}}{r^2}\frac{v_1}{r}\right) 4\pi r^2\, dr$$

$$= \frac{4\pi A\psi_0 Q\, b_0^{3-n} b^{2n}}{3q}\left(\frac{1}{a^n} - \frac{1}{b^n}\right)$$

where use has been made of (23), (24) and (30).

The equality of $\frac{dE}{d\rho}$ and W^p yields

$$\frac{P_0}{a_0}(a^3 - a_0^3) = \frac{A\psi_0 Q\, b_0^{3-n} b^n}{3q}\left[\left(\frac{b}{a}\right)^n - 1\right] \tag{32}$$

Using (6), we write

$$\left.\begin{aligned}\left(\frac{b}{a}\right)^n - 1 &= \frac{b^n}{(b^3 - k_0^3)^{n/3}} - 1 = \left(1 - \frac{k_0^3}{b^3}\right)^{-n/3} - 1\\ &\simeq \frac{nk_0^3}{3b^3}\end{aligned}\right\} \tag{33}$$

where second and higher powers of k_0^3/b^3 are omitted, since $k_0^3 < b^3$. Rearrangement of (6) gives

$$a^3 - a_0^3 = b^3 - b_0^3 \tag{34}$$

From (32), (33) and (34), we obtain

$$b^6 - b_0^3 b^3 - \lambda b^n = 0 \tag{35}$$

where

$$\lambda = \frac{AQ\psi_0\, a_0\, b_0^{3-n}\, nk_0^3}{9 P_0 q}$$

is a *large* quantity, and $0 < n < 2$.

It is easily verified that for known values of b_0, n and λ, the (35) has a real root, say $c > b_0$ giving the outer radius of the spherical shell at the instant of fracture.

It is generally a hazardous task to extract the roots of the (35). However, a lower bound for b can be constructed by the following quasi-linearization technique and thereafter numerical methods can be adopted to find the best approximation for b.

Dividing by $b_0^3\, b^6$, the (35) reduces to

$$\left.\begin{array}{ll} & x + \beta x^m = \alpha \\ \text{where} & x = \frac{1}{b^3}\,;\ \alpha = \frac{1}{b_0^3} > 0\,;\ \beta = \frac{\lambda}{b_0^3} > 0 \\ \text{and} & 2 > m = \frac{6-n}{3} > 1 \end{array}\right\} \quad (36)$$

A positive solution can be represented in the analytic form[5]

$$x = \min_{y} \left[\frac{\alpha + \beta (m-1)\, y^m}{1 + \beta m y^{m-1}} \right] \quad (37)$$

or, equivalently

$$b = \left[\max_{y} \left\{ \frac{1 + \beta m y^{m-1}}{\alpha + \beta (m-1)\, y^m} \right\} \right]^{\frac{1}{3}} \quad (38)$$

if we recall that $= \frac{1}{b^3}$

Thus

$$b \geqslant \left[\frac{1 + \beta m y^{m-1}}{\alpha + \beta (m-1)\, y^m} \right] = c_0 \text{ (say)} \quad (39)$$

for any value of $y \neq \frac{1}{b^3}$ and the equality is attained for $y = \frac{1}{b^3}$.

Once the values of α, β and m are known for the material under consideration, the appropriate choice of y and hence the lower bound for b namely c_0 (which is known to be greater than b_0) can be easily calculated.

Rewriting (36) in the form $H(x) \equiv x + \beta x^m - \alpha = 0$ (40)

it is observed that Newton-Raphson iteration method may be adopted by taking $\frac{1}{c_0^3}$ as the initial approximation for x in (40) [since the sufficient conditions for convergence of this process are seen to be satisfied by the function $H(x)$] and hence the value of b, the outer radius of the shell at the instant of fracture may be obtained for sufficient degree of accuracy.

When the value of b thus obtained is substituted in (22) through (25) and (30), the stress and velocity fields during plastic flow as also the critical pressure that causes fracture of the shell are completely determined.

As an application of the above theory in Defence Science and Ordnance Factories, we can cite the generation of a shock wave by an explosion of charges in the form of spherical, cylindrical and rectangular blocks. The symmetry of the spherical charge makes it the easiest type of charge to analyse from the point of view of hock wave configuration as the explosion products expand radially. The calculation of critical pressure under which the spherical shell fractures can be utilised to select the most appropriate metal or alloy for manufacturing shells which when fired can cause maximum amount of destruction by splinters. Also proper mixing of explosives can be effected so as to have the optimum critical pressure.

By taking cylindrical co-ordinate system, the analysis may be re-constructed for use in design and manufacture of suitable gun barrels where autofrettage principle is employed.

REFERENCES

1. THOMAS, T. Y., *J. of Mathematics and Mechanics*, **19** (1969), 379–402.
2. THOMAS, T.Y., 'Plastic Flow and Fracture in Solids' (Academic Press, New York), 1961, p. 70 & 71.
3. CHADWICK, P., COX, A. D. & HOPKINS, H.G., 'Mechanics of Deep Underground Explosion' (Philosophical Transactions of the Royal Society of London), No. 1070, Vol. 256, 1964, p. 262 & 263.
4. HOFFMAN, O. & SACHS, G., 'Introduction to the Theory of Plasticity for Engineers' (Mc Graw Hill, USA.), 1953, p. 76.
5. RICHARD E. BELLMAN & ROBERT E. KALABA, 'Quasilinearization and Non-linear Boundary-Value Problems' (American Elsevier Publishing Company, New York), 1965, p. 13 & 14.