MAGNETOHYDRODYNAMIC BOUNDARY LAYER WITH PRESSURE GRADIENT AND SUCTION

G. NATH

Indian Institute of Science, Bangalore

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The combined effects of suction and magnetodynamic pressure gradient on a viscous steady incompressible electrically conducting fluid of constant properties near the leading edge of a porous semi-infinite plate with an applied magnetic field have been studied. The two-point boundary value problem governed by coupled nonlinear ordinary differential equations has been solved numerically using a method of solution that is based on a least-squares convergence criterion which leads to the unique solution of the problem. The results show that the skin friction and the tangential component of the magnetic field increase with the suction velocity and the magnetodynamic pressure gradient.

The flow of a viscous electrically conducting fluid of constant properties in the absence of mass transfer and magnetodynamic pressure gradient past a semi-infinite plate with applied magnetic field has been studied by various authors[1–3]. The effect of magnetodynamic pressure gradient on the above problem has been considered by Davies[4] and Gribben[5] who obtained approximate closed form solutions of the governing equations using respectively an iterative procedure and a series expansion method. It has been pointed out by Gribben[5] that one of the boundary conditions used by Davies[4] is physically unrealistic. All these authors have found that the flow is possible only when the parameter S (which is the square of the ratio of the Alfven speed to the free stream velocity) is less than unity. It has been also shown[6] that for $\beta = 0$, $0<\lambda<1$, and $0\leqslant S\leqslant S_{max}$ where β is the magnetodynamic pressure gradient parameter and λ is the magnetic Prandtl number, there exist two solutions, one corresponding to large value of the skin friction parameter $F''(0)$ and the another corresponding to the small values of $F''(0)$. The second solution corresponds to a thicker boundary layer[6]. Within the knowledge of the author, the effect of suction (or injection) with or without magnetodynamic pressure gradient on the above problem has not been reported in the available published literature.

The aim of the present analysis is to study the combined effects of suction and magnetodynamic pressure gradient on the above problem[4] under the assumption that the boundary layer is thin using the correct boundary conditions, and also to provide exact values of the skin friction parameter and the induced magnetic field on the surface of the plate in tabular form by solving the two-point boundary value problem governed by nonlinear differential equations numerically employing the method developed by Nachtsheim & Swigert[7] which ensures the uniqueness of the solution.

One of the important features of this method is that the asymptotic boundary conditions are satisfied at the edge of the boundary layer by adjusting the initial conditions so that the mean square error between the computed values and the asymptotic values is minimized. Convergence to the solution is rapid and appears to be insensitive to the initial guess values of the initial conditions. It may be noted that the use of the least-square convergence criterion leads to the unique solution[7]. Another advantage of this method is that the two main problems of integrating the boundary layer equations, namely, approximating the missing initial conditions and determining when to stop the integration are reduced to an automatic initial-value technique that can be easily programmed on high-speed computers. This method is capable of yielding solutions even for those two-point boundary value problems which are particularly sensitive to the initial conditions and where the method of shoot and hunt (which is one of the most common methods for solving these problems) fails. The computer time required for the solution is much less compared to that for other numerical methods.

GOVERNING EQUATIONS

We consider the steady two-dimensional flow of a viscous incompressible electrically conducting fluid of constant properties past a porous semi-infinite plate under the influence of a magnetodynamic pressure gradient. The applied magnetic field is taken as uniform and parallel to the plate. The plate is considered to be unmagnetized and non-conducting. We assume that the viscous Reynolds number and the magnetic Reynolds number are sufficiently large for momentum and magnetic boundary layers to have developed[4].

Under these assumptions, using boundary layer approximations, the governing equations in non-dimensional form under similarity requirements, taking into account the effect of suction, can be expressed as[4–5]

$$F''' + F F'' + \beta (1 - F'^2) = S [G G'' + \beta (1 - G^2)] \tag{1}$$

$$G''' + \lambda (F G'' - F'' G) = 0 \tag{2}$$

with boundary conditions

$$\left.\begin{aligned} &F(0) = F_w,\ F'(0) = 0,\ F'(\infty) \to 1 \\ &G(0) = G''(0) = 0,\ G'(\infty) \to 1 \end{aligned}\right\} \tag{3}$$

(The notations are the same as those employed in reference 4). Here F and G are respectively nondimensional velocity and magnetic stream functions depending on the similarity variable η only, S is the square of the ratio of the Alfven speed to the free stream velocity, β is the magnetodynamic pressure gradient parameter, λ is the magnetic Prandtl number, F_w ($F_w > 0$) is the suction parameter and the prime denotes differentiation with respect to η. The suction velocity v has been assumed to vary as $x^{(n+1)/4}$ to satisfy the similarity requirements (x is the distance along the plate measured from its leading edge and n is a constant).

The quantities of chief practical interest are the skin friction and the tangential component of the magnetic field and they are expressed as[4]

$$\tau_w = \mu U^{3/2} [(2-\beta)\nu]^{-1/2}\ x^{(2\beta-1)/(2-\beta)}\ F''(0)$$

$$H_1 = H x^{\beta/(2-\beta)}\ G'(0) \tag{4}$$

where τ_w is the shear stress at the wall, μ is the viscosity, ν is the kinematic viscosity, U is the free-stream velocity, H is the magnetic field and H_1 is the tangential component of the magnetic field.

NUMERICAL SOLUTION

Equations (1) to (3) represent a nonlinear two-point boundary value problem and they were solved by specifying two additional conditions at the surface (i.e., at $\eta = 0$) and integrating the equations as an initial value problem. The values of the additional conditions were adjusted till the asymptotic boundary conditions were satisfied. From the detailed description of the method given by Nachtsheim & Swigert[7], and following their procedure, we define two additional initial conditions as variables

$$F''(0) = a,\ G'(0) = b \tag{5}$$

and their values are adjusted such that

$$F'(\infty) \to 1 \text{ and } G'(\infty) \to 1 \tag{6}$$

Using this iterative numerical procedure, we determine the corrections $\triangle a$ and $\triangle b$ for the next iteration in the least square sense from the solution of the following equations

$$F' + F'_a \triangle a + F'_b \triangle b = 1 \tag{7}$$

$$G' + G'_a \triangle a + G'_b \triangle b = 1 \tag{8}$$

$$F'' + F''_a \triangle a + F''_b \triangle b = 0 \tag{9}$$

$$G'' + G''_a \triangle a + G''_b \triangle b = 0 \tag{10}$$

at the edge of the boundary layer. The subscripts a and b denote partial differentiation with respect to a and b respectively. The conditions (9) and (10) ensure that the boundary conditions are satisfied asymptotically. The corrections $\triangle a$ and $\triangle b$ are obtained from the matrix equation.

$$\begin{bmatrix} F'^2_a + G'^2_a + F''^2_a + G''^2_a & F'_a F'_b + G'_a G'_b + F''_a F''_b + G''_a G''_b \\ F'_a F'_b + G'_a G'_b + F''_a F''_b + G''_a G''_b & F'^2_b + G'^2_b + F''^2_b + G''^2_b \end{bmatrix} \times$$

$$\times \begin{bmatrix} \triangle a \\ \triangle b \end{bmatrix} = - \begin{bmatrix} F' F'_a + G' G'_a + F'' F''_a + G'' G''_a - F'_a - G'_a \\ F' F'_b + G' G'_b + F'' F''_b + G'' G''_b - F'_b - G'_b \end{bmatrix} \tag{11}$$

The partial derivatives of F and G with respect to a and b in (11) are obtained by integration of two sets of auxiliary equations formed by differentiating equations (1-2) with respect to a and b. The set of equations for the a-derivatives are given by

$$F''' = -(F'' F_a + F F''_a - 2\beta F' F'_a) + S(G'' G_a + G G''_a - 2\beta G G_a) \tag{12}$$

$$G'''_a = \lambda (G F''_a + F'' G_a - G'' F_a - F G''_a) \tag{13}$$

The initial conditions for the above set of equations are obtained by differentiating (3) and (5) with respect to a and they are expressed as

$$F_a(0) = F'_a(0) = 0,\ F''_a(0) = 1 \qquad G_a(0) = G'_a(0) = G''_a(0) = 0 \tag{14}$$

The set of auxiliary equations for the b-derivatives are identical to (12) and (13) when a is replaced by b and the initial conditions for this set of equations are given by

$$F_b(0) = F'_b(0) = F''_b(0) = 0 \qquad G_b(0) = G''_b(0) = 0,\ G'_b(0) = 1 \tag{15}$$

Equations (1) and (2) alongwith their auxiliary equations are numerically integrated to some arbitrary value of $\eta = \eta_{stop}$. The corrections $\triangle a$ and $\triangle b$ are then obtained from the solutions of (11) and the new values of a and b are

$$a_{new} = a_{old} + \triangle a\ ; \qquad b_{new} = b_{old} + \triangle b \tag{16}$$

This iterative procedure of integration is repeated till the condition

$$(\,|\triangle a| + |\triangle b|\,)_{max} \leqslant 10^{-4} \tag{17}$$

is satisfied. Then we obtain the minimum value of the error term E i.e. E_{min} at η_{stop} from the expression

$$E = (1 - F')^2 + (1 - G')^2 + F''^2 + G''^2 \tag{18}$$

where E_{min} corresponds to those values of a and b for which $\triangle a$ and $\triangle b$ are zero. If E_{min} is not less than a certain desired value say 10^{-5}, the range of integration η_{stop} is increased. The entire process described above is repeated until the asymptotic boundary conditions are satisfied to the desired degree of accuracy. It may be remarked that as η_{stop} is increased, E_{min} tends to zero and the required initial conditions $a = F''(0)$ and $b = G'(0)$ also tend to definite values. Thus E_{min} can be used as a stopping criterion for terminating the computer calculation and for finding η_{edge} automatically.

RESULTS AND DISCUSSION

The integration of (1) and (2) alongwith their auxiliary equations was carried out on IBM 360 digital computer with a predictor-corrector (Adams-Moulton) subroutine[7–8] with step size $\triangle\eta = 0{\cdot}1$ for some representative values of the parameters F_w, β, S and λ. It was found that η_{edge} depends upon the values of the parameters, but it is more sensitive to λ. The velocity and the tangential magnetic field profiles are shown in Figs. 1-2 and the surface skin friction parameter $F''(0)$ and the tangential component of the magnetic field at the surface, $G'(0)$ are given in Table 1.

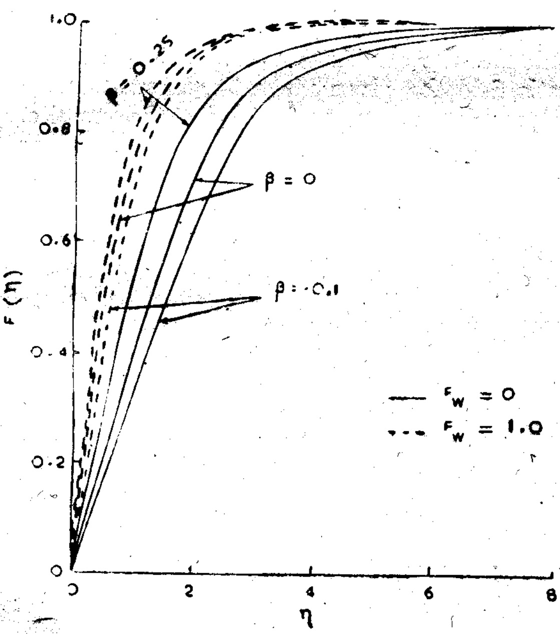

Fig. 1—Velocity distribution for $S = 0{\cdot}25$ and $\lambda = 0{\cdot}2$.

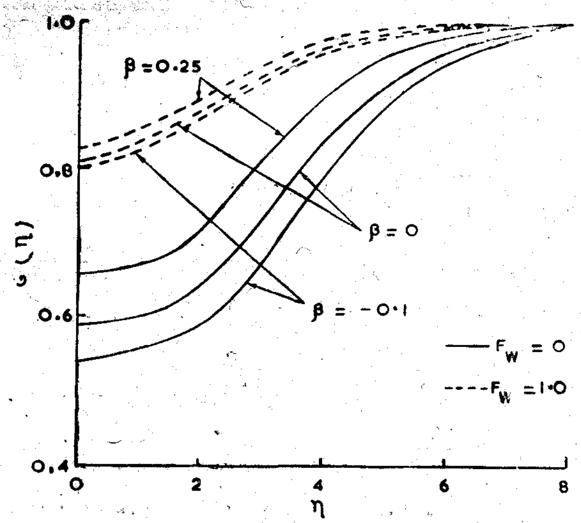

Fig. 2—Magnetic field distribution for $S = 0{\cdot}25$ and $\lambda = 0{\cdot}2$.

TABLE 1

SKIN FRICTION PARAMETER, $F''(0)$, AND THE TANGENTIAL COMPONENT OF THE MAGNETIC FIELD $G'(0)$

S	β	λ	$F_w=0$		$F_w=1$		$F_w=2$	
			$F''(0)$	$G''(0)$	$F''(0)$	$G''(0)$	$F''(0)$	$G''(0)$
0·05	0	0·1	0·4591	0·6929	1·2757	0·8439	2·1894	0·9092
0·05	0	0·2	0·4575	0·6230	1·2749	0·8106	2·1893	0·8932
0·05	0·125	0·1	0·6018	0·7229	1·3704	0·8504	2·2571	0·9114
0·05	0·125	0·2	0·5999	0·6565	1·3695	0·8183	2·2572	0·8957
0·05	0·25	0·2	0·7173	0·6802	1·4562	0·8252	2·3215	0·8982
0·05	—0·10	0·1	0·3091	0·6542	1·1927	0·8370	2·1326	0·9073
0·05	—0·10	0·2	0·3078	0·5803	1·1908	0·8030	2·1325	0·8911
0·125	0	0·1	0·4424	0·6824	1·2632	0·8404	2·1811	0·9078
0·125	0	0·2	0·4383	0·6115	1·2612	0·8069	2·1810	0·8920
0·125	0·125	0·1	0·5837	0·7136	1·3563	0·8468	2·2475	0·9100
0·125	0·125	0·2	0·5787	0·6461	1·3538	0·8146	2·2473	0·8944
0·125	0·25	0·2	0·6943	0·6708	1·4391	0·8216	2·3105	0·8969
0·125	—0·1	0·1	0·2937	0·6413	1·1818	0·8343	2·1257	0·9064
0·125	—0·1	0·2	0·2897	0·5668	1·1792	0·7995	2·1255	0·8899
0·25	0	0·1	0·4115	0·6623	1·2407	0·8338	2·1663	0·9053
0·25	0	0·2	0·4033	0·5901	1·2368	0·8003	2·1663	0·8899
0·25	0·125	0·1	0·5505	0·6958	1·3307	0·8404	2·2306	0·9075
0·25	0·125	0·2	0·5403	0·6266	1·3258	0·8079	2·2302	0·8922
0·25	0·25	0·2	0·6523	0·6519	1·4083	0·8155	2·2909	0·8947
0·25	—0·1	0·1	0·2628	0·6167	1·1629	0·8279	2·1143	0·9042
0·25	—0·1	0·2	0·2565	0·5414	1·1592	0·7931	2·1128	0·8879

The results show that $F''(0)$ and $G'(0)$ increase with F_w and β for given S and λ. For prescribed F_w, S and β, $F''(0)$ changes little with λ and similar behaviour is observed for $G'(0)$ when F_w is large. For $F_w = 0$, $G'(0)$ decreases with increase in λ. Similarly, for given F_w, β and λ, $F''(0)$ decreases as S increases. However, the effect of S on $G'(0)$ is similar to that of λ.

CONCLUSIONS

The effect of the suction velocity and the magnetodynamic pressure gradient (favourable) is to increase the skin friction and the tangential component of the magnetic field for given magnetic parameter and magnetic Prandtl number. The effect of adverse magnetodynamic pressure gradient is just the opposite. The surface skin friction seems to be little affected by the change in the magnetic Prandtl number. Similar behaviour is observed for the tangential component of the magnetic field for large suction velocity. The present analysis is another indication of the accuracy, reliability and utility of the method which can also be used successfully for the solutions of boundary layer type of equations describing more complex flow fields.

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