UNSYMMETRICAL CRUCIFORM CRACK IN A THIN CIRCULAR ELASTIC PLATE

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The problem of determining stress distribution in a thin circular elastic plate containing unsymmetrical cruciform crack is considered. The faces of the crack are subjected to different internal pressures. The problem is reduced to a pair of dual integral equations. These equations are further reduced to simultaneous Fredholm integral equations of the second kind which are solved numerically. Quantities of physical interest are calculated.

Stallybrass[1], Rooke and Sneddon[2] have considered the problem of determining the stress distribution in an elastic solid containing a crack in the form of a cross, with arms of equal length. Stallybrass[3] also considered the same problem and obtained explicit formulae for an arbitrary, but integrable, distribution of pressure. Srivastava and Gupta[4] have solved the problem of determining the stress distribution in a thin elastic circular plate containing a cruciform crack of equal arm lengths.

Recently, Sneddon and Das[5] have considered the problem of determining the distribution of stress in an infinite medium containing unsymmetrical cruciform crack. In this paper we consider the problem of finding the stress distribution in a thin circular elastic plate containing unsymmetrical cruciform crack. The corresponding mixed boundary value problem is solved by reducing it to a pair of dual integral equations. This pair is further reduced to simultaneous Fredholm integral equations of the second kind which are solved numerically.

The present investigation are motivated with the object of finding perturbation in the stress and displacement field due to finite circular boundaries.

BASIC EQUATIONS

We superimpose the expressions of the circular plate problem given by Srivastava and Kumar[6] on the expressions of the non-vanishing stress tensors ($\sigma_{\theta\theta}$, σ_{rr}, $\sigma_{r\theta}$) and displacement vector (u_θ, u_r, o) given[5] and obtain the following expressions :

$$\sigma_{\theta\theta}(r,\theta) = -\int_0^\infty B(\xi)\, e^{-\xi (r/a)\sin\theta} \Big[\xi (r/a) \sin\theta \cdot \cos\{2\theta + \xi (r/a)\cos\theta\} + \cos\{\xi (r/a)\cos\theta)\}\Big] d\xi +$$

$$+ \int_0^\infty A(\xi)\, e^{-\xi (r/b)\cos\theta} \Big[\xi (r/b)\cos\theta \cos\{2\theta - \xi (r/b)\sin\theta\} - \cos\{\xi (r/b)\sin\theta\}\Big] d\xi +$$

$$+ \sum_{n=o}^{\infty} \Big[n(n-1)\, c_n\, r^{n-2} + (n+1)(n+2)\, d_n\, r^n \Big] \cos n\theta \qquad (1)$$

$$\sigma_{rr}(r,\theta) = -\int_0^\infty B(\xi)\, e^{-\xi (r/a)\sin\theta} \Big[\cos\{\xi (r/a)\cos\theta\} - \xi (r/a)\sin\theta \cos\{2\theta + \xi (r/a)\cos\theta\}\Big] d\xi -$$

$$- \int_0^\infty A(\xi)\, e^{-\xi (r/b) \cos\theta} \Big[\cos\{\xi (r/b) \sin\theta\} +$$

$$+ \xi (r/b) \cos\theta \cos\{2\theta - \xi (r/b) \sin\theta\} \Big] d\xi +$$

$$+ \sum_{n=o}^{\infty} \Big[n(n-1)\, c_n r^{n-2} + (n+1)(n-2)\, d_n r^n \Big] \cos n\theta \qquad (2)$$

$$\tau_{r\theta}(r, \theta) = - \int_0^\infty B(\xi)\, e^{-\xi (r/a) \sin\theta} \Big[\xi (r/a) \sin\theta \sin\{2\theta +$$

$$+ \xi (r/a) \cos\theta\} \Big] d\xi + \int_0^\infty A(\xi)\, e^{-\xi (r/b) \cos\theta} \Big[\xi (r/b)$$

$$\cdot \cos\theta \sin\{2\theta - \xi (r/b) \sin\theta\} \Big] d\xi -$$

$$- \sum_{n=o}^{\infty} \Big[n(n-1)\, c_n r^{n-2} + n(n+1)\, d_n r^n \Big] \sin n\theta \qquad (3)$$

$$2\mu u_\theta(r, \theta) = a \int_0^\infty \xi^{-1} B(\xi)\, e^{-\xi (r/a) \sin\theta} \Big[\{2 - 2\eta +$$

$$+ \xi (r/a) \sin\theta\} \cos\theta \cos\{\xi (r/a) \cos\theta\} + \{1 - 2\eta -$$

$$- \xi (r/a) \sin\theta\} \sin\theta \sin\{\xi (r/a) \cos\theta\} \Big] d\xi -$$

$$- b \int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi (r/b) \cos\theta} \Big[\{1 - 2\eta - \xi (r/b) \cos\theta\} \cdot$$

$$\cdot \cos\theta \sin\{\xi (r/b) \sin\theta\} + \{2 - 2\eta + \xi (r/b) \cos\theta\} \cdot$$

$$\cdot \sin\theta \cos\{\xi (r/b) \sin\theta\} \Big] d\xi + \sum_{n=0}^{\infty} \Big[n\, c_n r^{n-1} + (n -$$

$$- 4\eta + 4)\, d_n r^{n+1} \Big] \sin n\theta \qquad (4)$$

$$2\mu u_r(r, \theta) = a \int_0^\infty \xi^{-1} B(\xi)\, e^{-\xi (r/a) \sin\theta} \Big[\Big\{ 2 - 2\eta +$$

$$+ \xi (r/a) \sin\theta \Big\} \sin\theta \cos\Big\{ \xi (r/a) \cos\theta \Big\} -$$

$$- \Big\{ 1 - 2\eta - \xi (r/a) \sin\theta \Big\} \cos\theta \sin\Big\{ \xi (r/a) \cos\theta \Big\} \Big] d\xi +$$

$$+ b \int_0^\infty \xi^{-1} A(\xi)\, e^{-\xi (r/b) \cos\theta} \Big[\Big\{ 2 - 2\eta + \xi (r/b) \cos\theta \Big\} \cdot$$

$$\cdot \cos\theta \cos\left\{\xi (r/b) \sin\theta\right\} - \left\{1 - 2\eta - \xi (r/b) \cos\theta\right\} \sin\theta \cdot$$

$$\cdot \sin\left\{\xi (r/b) \sin\theta\right\}\Big] d\xi - \sum_{n=0}^{\infty} \Big[n c_n r^{n-1} +$$

$$+ (n - 4\eta - 2) d_n r^{n+1}\Big] \cos n\theta \tag{5}$$

where μ and η are modulus of rigidity and Poisson's ratio for the material of the plate respectively.

FORMULATION OF THE PROBLEM

We consider, under assumption of plane strain, a thin circular elastic plate containing unsymmetrical cruciform crack. In plane polar co-ordinates (r, θ), the plate is supposed to occupy the region $0 < r < \rho$, $0 < \theta < 2\pi$, and the crack inside the plate occupies the region $0 < r < a$, $\theta = 0, \pi$; $0 < r < b$, $\theta = \pi/2$, $3\pi/2$, where ρ is the radius of the plate and a and b are length of the two arms of the cracks. The faces of the crack in the arm $0 < r < a$, $\theta = 0, \pi$ are subjected to pressure $f(r/a)$ and those on $0 < r < b$, $\theta = \pi/2$, $3\pi/2$; are subjected to pressure $g(r/b)$. It is further assumed that these are even functions of r. Thus the problem reduces to a quarter plate problem. The boundary conditions on $\theta = 0$ are

$$\left.\begin{aligned} &\sigma_{\theta\theta}(r, 0) = -f(r/a); && 0 < r < a, && (a)\\ &u_\theta(r, 0) = 0; && a < r < \rho, && (b)\\ &\sigma_{r\theta}(r, 0) = 0; && 0 \leq r < \rho, && (c)\end{aligned}\right\} \tag{6}$$

The boundary conditions on $\theta = \pi/2$ are

$$\left.\begin{aligned} &\sigma_{\theta\theta}(r, \pi/2) = -g(r/b); && 0 < r < b, && (a)\\ &u_\theta(r, \pi/2) = 0; && b < r < \rho, && (b)\\ &\sigma_{r\theta}(r, \pi/2) = 0; && 0 < r < \rho, && (c)\end{aligned}\right\} \tag{7}$$

and conditions on free boundary are

$$\sigma_{r\theta}(\rho, \theta) = \sigma_{rr}(\rho, \theta) = 0; \; 0 < \theta < \pi/2 \tag{8}$$

DERIVATION OF INTEGRAL EQUATIONS

The boundary conditions (6 c) and (7c) are automatically satisfied if all odd-coefficients c_{2n+1} and d_{2n+1} are zero. The boundary conditions (6 a) and (6 b) & (7 a) and (7 b) lead to a pair of dual integral equations :

$$\frac{d}{dr}\left[a \int_0^{\infty} \xi^{-1} B(\xi) \sin\xi (r/a)\, d\xi + r \int_0^{\infty} A(\xi)\, e^{-\xi (r/b)}\, d\xi \right] = F_1(r/a); \quad 0 < r < a,$$

$$\int_0^{\infty} \xi^{-1} B(\xi) \cos\xi (r/a)\, d\xi = 0; \quad a < r < \rho,$$

Substituting $r/a = v$ and $a/b = \alpha$,

we get

$$\frac{d}{dv}\left[\int_0^{\infty} \xi^{-1} B(\xi) \sin\xi v\, d\xi + v \int_0^{\infty} A(\xi)\, e^{-\xi \alpha v}\, d\xi \right] = F(v); \quad 0 < v < 1, \tag{9}$$

$$\int_0^{\infty} \xi^{-1} B(\xi) \cos\xi v\, d\xi = 0; \quad 1 < v < \rho, \tag{10}$$

and

$$\frac{d}{dr}\left[b\int_0^\infty \xi^{-1} A(\xi)\sin\xi(r/b)\,d\xi + r\int_0^\infty B(\xi)\,e^{-\xi(r/a)}\,d\xi\right] = G(r/b)\,;\quad 0<r<b\,,$$

$$\int_0^\infty \xi^{-1} A(\xi)\cos\xi(r/b)\,d\xi = 0\,;\quad b<r<\rho\,.$$

Substituting $r/b = w$ and $b/a = \beta = \alpha^{-1}$,
we get

$$\frac{d}{dw}\left[\int_0^\infty \xi^{-1} A(\xi)\sin\xi w\,d\xi + \int_0^\infty B(\xi)\,e^{-\xi\beta w}\,d\xi\right] = G(w)\,;\quad 0<w<1\,, \tag{11}$$

$$\int_0^\infty \xi^{-1} A(\xi)\cos\xi w\,d\xi = 0\,;\quad 1<w<\rho\,. \tag{12}$$

where

$$F(v) = f(v) + \sum_{n=0}^{\infty}(2n+1)(2n+2)(c_{2n+2}+d_{2n})\,a^{2n}v^{2n} \tag{13}$$

and

$$G(w) = g(w) + \sum_{n=0}^{\infty}(-1)^n(2n+1)(2n+2)(c_{2n+2}+d_{2n})\,b^{2n}w^{2n} \tag{14}$$

The solution of these dual integral equations as given by Sneddon[7] are

$$A(\xi) = \xi\int_0^1 t\,a(t)\,J_0(\xi t)\,dt\,, \tag{15}$$

$$B(\xi) = \xi\int_0^1 t\,b(t)\,J_0(\xi t)\,dt\,, \tag{16}$$

where $a(t)$ and $b(t)$ are the unknown functions to be determined later.

With this choice of $A(\xi)$ and $B(\xi)$ the equations (10) and (12) are automatically satisfied, while the equations (9) and (11) lead to

$$\frac{d}{dv}\left[\int_0^v \frac{t\,b(t)\,dt}{(v^2-t^2)^{\frac{1}{2}}}\right] + \frac{d}{dv}\left[\alpha v^2\int_0^1 \frac{t\,a(t)\,dt}{(t^2+\alpha^2/v^2)^{3/2}}\right] = F(v)\,;\quad 0<v<1\,,$$

$$\frac{d}{dw}\left[\int_0^w \frac{t\,a(t)\,dt}{(w^2-t^2)^{\frac{1}{2}}}\right] + \frac{d}{dw}\left[\beta w^2\int_0^1 \frac{t\,b(t)\,dt}{(t^2+\beta^2/w^2)^{3/2}}\right] = G(w)\,;\quad 0<w<1\,.$$

Inverting these Abel integral equations,
we get

$$b(t) + \int_0^1 K(\alpha t, u)\,a(u)\,du = \phi(t)\,;\quad 0<t<1\,, \tag{17}$$

and

$$a(t) + \int_0^1 K(\beta t, u)\, b(u)\, du = \psi(t)\,; \quad 0 < t < 1\,. \tag{18}$$

where

$$\phi(t) = \frac{2}{\pi}\int_0^t \frac{F(v)\,dv}{(t^2 - v^2)^{\frac{1}{2}}} = \frac{2}{\pi}\int_0^t \frac{f(v)\,dv}{(t^2 - v^2)^{\frac{1}{2}}} + $$

$$+ \sum_0^\infty \frac{(2n+1)(2n+2)\left(\frac{1}{2}\right)_n}{\lfloor n} (c_{2n+2} + d_{2n})\, a^{2n} t^{2n}, \tag{19}$$

$$\psi(t) = \frac{2}{\pi}\int_0^t \frac{G(w)\,dw}{(t^2 - w^2)^{\frac{1}{2}}} = \frac{2}{\pi}\int_0^t \frac{g(w)\,dw}{(t^2 - w^2)^{\frac{1}{2}}} + $$

$$+ \sum_{n=0}^\infty \frac{(-1)^n (2n+1)(2n+2)\left(\frac{1}{2}\right)_n}{\lfloor n} (c_{2n+2} + d_{2n})\, b^{2n} t^{2n} \tag{20}$$

$$K(\alpha t, u) = \frac{2\alpha u}{\pi t}\frac{d}{dt}\int_0^t \frac{s^3\,ds}{(t^2 - s^2)^{\frac{1}{2}}(u^2 + \alpha^2 s^2)^{3/2}} = \frac{4\alpha t u^2}{\pi(u^2 + \alpha^2 t^2)^2}\,; \tag{21}$$

and

$$K(\beta t, u) = 4\beta t u^2 / \pi (u^2 + \beta^2 t^2)^2\,. \tag{22}$$

REDUCTION TO SIMULTANEOUS FREDHOLM INTEGRAL EQUATIONS

The condition (8) is satisfied provided

$$-\int_0^\infty B(\xi)\, e^{-\xi(\rho/a)\sin\theta}\left[\cos\left\{\xi(\rho/a)\cos\theta\right\} - \xi(\rho/a)\cos\left\{2\theta + \xi(\rho/a)\cos\theta\right\}\sin\theta\right] d\xi -$$

$$-\int_0^\infty A(\xi)\, e^{-\xi(\rho/b)\cos\theta}\left[\cos\left\{\xi(\rho/b)\sin\theta\right\} + \xi(\rho/b)\cos\theta\cos\left\{2\theta + \xi(\rho/b)\sin\theta\right\}\right] d\xi -$$

$$-\sum_{n=0}^\infty\left[2n(2n-1)\,c_{2n}\,\rho^{2n-2} + (2n+1)(2n-2)\,d_{2n}\,\rho^{2n}\right]\cos 2n\theta = 0\,;\ 0 < \theta < \pi/2 \tag{23}$$

and

$$-\int_0^\infty B(\xi)\, e^{-\xi(\rho/a)\sin\theta}\left[\xi(\rho/a)\sin\theta\sin\left\{2\theta + \xi(\rho/a)\cos\theta\right\}\right] d\xi + \int_0^\infty A(\xi)\, e^{-\xi(\rho/b)\cos\theta}\,.$$

$$\cdot\left[\xi(\rho/b)\cos\theta\sin\left\{2\theta - \xi(\rho/b)\sin\theta\right\}\right] d\xi - \sum_{n=0}^\infty\left[2n(2n-1)\,c_{2n}\,\rho^{2n-2} + \right.$$

$$\left. + d_{2n}\,\rho^{2n}\right]\sin 2n\theta = 0\,;\quad 0 < \theta < \pi/2 \tag{24}$$

Substituting the values of $A(\xi)$ and $B(\xi)$ in the above expressions ; changing order of integration and using the results which are given in appendix, we find that

$$\frac{(A_0 + B_0)}{\rho^2} - 2\, d_0 = 0, \tag{25}$$

For $n \geqslant 1$

$$(\tfrac{1}{2})_n \frac{(n+1)}{\underline{|n}\, \rho^{2n}} \left[2n\,(A_{n-1} - B_{n-1}) + \frac{(2n+1)(A_n + B_n)}{\rho^2} \right] +$$
$$+ \left[2n\,(2n-1)\, c_{2n}\, \rho^{2n-2} + (2n+1)(2n-2)\, d_{2n}\, \rho^{2n} \right] = 0\,, \tag{26}$$

$$\frac{(\frac{1}{2})_n}{\underline{|n}\, \rho^{2n}} \left[2n^2 (A_{n-1} - B_{n-1}) + \frac{(2n+1)(n+1)}{\rho^2} (A_n + B_n) \right] -$$
$$- \left[2n \left\{ (2n-1)\, c_{2n}\, \rho^{2n-2} + (2n+1)\, d_{2n}\, \rho^{2n} \right\} \right] = 0\,, \tag{27}$$

From these expressions, we get

$$d_0 = (A_0 + B_0)/2\,\rho^2 \tag{28}$$

and

$$d_{2n} = \frac{(\frac{1}{2})_n}{\underline{|n}\, \rho^{4n}} \left[n\,(A_{n-1} - B_{n-1}) + \frac{(n+1)(A_n + B_n)}{\rho^2} \right] \tag{29}$$

$$c_{2n} = -\frac{(\frac{1}{2})_n}{\underline{|n}\, \rho^{4n-2}} \left[\frac{2n^2}{(2n-1)} (A_{n-1} - B_{n-1}) + \frac{(2n+1)(n+1)}{2n\,\rho^2} (A_n + B_n) \right]$$

i.e.

$$c_{2n+2} = -\frac{(\frac{1}{2})_{n+1}}{\underline{|(n+1)}\, \rho^{4n+2}} \left[\frac{2(n+1)^2}{(2n+1)} (A_n - B_n) + \frac{(2n+3)(n+2)}{(2n+2)\,\rho^2} (A_{n+1} + B_{n+1}) \right] \tag{30}$$

where

$$A_n = (-1)^n \int_0^1 b^{2n+2}\, a(t)\, t^{2n+1}\, dt\,,$$

$$B_n = \int_0^1 a^{2n+2}\, b(t)\, t^{2n+1}\, dt\,.$$

Substituting the unknown coefficients in (17) and (18), we get

$$b(t) + \int_0^1 \left[K(\alpha t, u) + K_1(t, u) \right] a(u)\, du + \int_0^1 K_2(t, u)\, b(u)\, du = \frac{2}{\pi} \int_0^1 \frac{f(v)}{(t^2 - v^2)^{\frac{1}{2}}}\, dv \tag{31}$$

and

$$a(t) + \int_0^1 \left[K(\beta t, u) + L_1(t, u) \right] b(u)\, du + \int_0^1 L_2(t, u)\, a(u)\, du = \frac{2}{\pi} \int_0^t \frac{g(w)}{(t^2 - w^2)^{\frac{1}{2}}}\, dw \tag{32}$$

where

$$K_1(t, u) = u\,(b/\rho)^2 - 3\,u^3 (b/\rho)^4 + \sum_{1}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\frac{1}{2})_n (2n+1)(2n+2)}{(\underline{n})^2} \cdot$$
$$\cdot \left(\frac{abut}{\rho^2} \right)^{2n} \frac{1}{u} \left[n - \frac{(2n+3)(2n+1)(n+2)}{4(n+1)^2} (bu/\rho)^4 \right],$$

$$K_2(t, u) = -3u(a/\rho)^2 + 3u^3(a/\rho)^4 + \sum_1^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n(2n+1)(2n+2)}{(\underline{|n})^2}$$

$$\left(\frac{a^2 u t}{\rho^2}\right)^{2n} \frac{1}{u}\left[n - (2n+2)(a u/\rho)^2 + \frac{(2n+3)(2n+1)(n+2)a^4u^4}{4(n+1)^2\rho^4}\right],$$

$$L_1(t, u) = -3u(a/\rho)^2 + 3u^3(a/\rho)^4 - \sum_1^{\infty} \frac{(-1)^n(\frac{1}{2})_n(\frac{1}{2})_n}{(\underline{|n})^2}(2n+2)(2n+1)$$

$$\cdot\left(\frac{abut}{\rho^2}\right)^{2n} \frac{1}{u}\left[-n + 2(n+1)(a u/\rho)^2 - \frac{(2n+3)(2n+1)(n+2)}{4(n+1)^2}(a u/\rho)^4\right],$$

$$L_2(t, u) = u(b/\rho)^2 - 3u^3(b/\rho)^4 + \sum_1^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n(2n+1)(2n+2)}{(\underline{|n})^2}$$

$$\left(\frac{b^2 u t}{\rho^2}\right)^{2n} \frac{1}{u}\left[n - \frac{(2n+3)(2n+1)(n+2)}{4(n+1)^2}(b u/\rho)^4\right].$$

The numerical solution of integral equations (31) and (32) is obtained.

QUANTITIES OF PHYSICAL INTEREST

The problem of determining the components of the displacement and stress fields therefore reduces to that of solving the pair of simultaneous integral equations (31) and (32) for the unknown function $a(t)$ and $b(t)$. We assume that the faces of the cracks are subjected to uniform internal pressure p_0.

The normal displacement along the line of cracks are

$$2\mu u_\theta(r, 0) = 2(1-\eta)\, a \int_0^{\infty} \xi^{-1} B(\xi) \cos(\xi (r/a))\, d\xi,$$

and

$$2\mu u_\theta(r, \pi/2) = -2(1-\eta)\, b \int_0^{\infty} \xi^{-1} A(\xi) \cos \xi (r/b)\, d\xi.$$

Substituting the values of $A(\xi)$ and $B(\xi)$ and changing the order of integration, we get

$$u_\theta(r, 0) = \frac{2(1-\eta^2)p_0 a}{E} \int_{r/a}^{1} \frac{t\, b(t)\, dt}{(t^2 - r^2/a^2)^{\frac{1}{2}}}, \tag{33}$$

and

$$u_\theta(r, \pi/2) = -\frac{2(1-\eta^2)p_0 b}{E} \int_{r/b}^{1} \frac{t\, a(t)\, dt}{(t^2 - r^2/b^2)^{\frac{1}{2}}}. \tag{34}$$

Thus we can easily find the normal displacements at the junction of the crack by setting $r = 0$ in (33) and (34). The total crack energy is given by

$$W = W_1 + W_2$$

where

$$W_1 = 2 \int_0^1 f(r/a)\, u_\theta (r, 0)\, dr = 2\, p_0 \int_0^1 u_\theta (r, 0)\, dr.$$

and

$$W_2 = 2 \int_0^1 g(r/b)\, u_\theta (r, \pi/2)\, dr = 2\, p_0 \int_0^1 u_\theta (r, \pi/2)\, dr.$$

Inserting the values of $u_\theta (r, 0)$ and $u_\theta (r, \pi/2)$ from (33) and (34) we obtain

$$W_1 = \frac{2\pi (1 - \eta^2)\, p_o^2\, a^2}{E} \int_0^1 t\, b(t)\, dt$$

$$W_2 = - \frac{2\pi (1 - \eta^2)\, p_o^2\, b^2}{E} \int_0^1 t\, a(t)\, dt.$$

Thus

$$W = \frac{2\pi (1 - \eta^2)\, p_o^2}{E} \left[a^2 \int_0^1 t\, b(t)\, dt - b^2 \int_0^1 t\, a(t)\, dt \right] \tag{35}$$

where E denotes Young's modulus of the plate.

From equation (1), we deduce

$$\sigma_{\theta\theta} (r, 0) = - p_o \frac{d}{dr} \left[a \int_0^\infty \xi^{-1} B(\xi) \sin \xi (r/a)\, d\xi + \right.$$

$$\left. + r \int_0^\infty A(\xi)\, e^{- \xi (r/b)}\, d\xi \right] - \sum_0^\infty (2n+1)(2n+2)(c_{2n+2} + d_{2n})\, r^{2n};\ r > a$$

$$\sigma_{\theta\theta} (r, \pi/2) = - p_o \frac{d}{dr} \left[b \int_0^\infty \xi^{-1} A(\xi) \sin \xi (r/b)\, d\xi + \right.$$

$$\left. + r \int_0^\infty B(\xi)\, e^{- \xi (r/a)}\, d\xi \right] - \sum_0^\infty (-1)^n (2n+1)(2n+2)(c_{2n+2} + d_{2n})\, r^{2n};\ r > b.$$

Substituting the values of $A(\xi)$ and $B(\xi)$ we find

$$\sigma_{\theta\theta}(r, 0) = \frac{p_o\, r}{(r^2 - a^2)^{\frac{1}{2}}}\, b(1) + O(1);\ r > a, \tag{36}$$

$$\sigma_{\theta\theta} (r, \pi/2) = \frac{p_o\, r}{(r^2 - b^2)^{\frac{1}{2}}}\, a(1) + O(1);\ r > b. \tag{37}$$

The stress intensity factors K, K^1 at the tips of the arms of the crack are given by

$$K = \underset{r \to a+}{Lt} (r-a)^{\frac{1}{2}} \left[\sigma_{\theta\theta} (r, 0) \right] = p_o \left(\tfrac{1}{2} a\right)^{\frac{1}{2}} b\,(1), \tag{38}$$

$$K' = \underset{r \to b+}{Lt} (r-b)^{\frac{1}{2}} \left[\sigma_{\theta\theta} (r, \pi/2) \right] = p_o \left(\tfrac{1}{2} b\right)^{\frac{1}{2}} a\,(1). \tag{39}$$

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

The integral equation (31) and (32) are numerically solved for $a(t)$ and $b(t)$ by the method of Fox and Goodwin[8]. The numerical values of $a(t)$ and $b(t)$ are calculated for various values of $\alpha = a/b$ ($= 0{\cdot}4$, $0{\cdot}5$, $0{\cdot}6$, $0{\cdot}7$, $0{\cdot}8$ and $0{\cdot}9$), and $\rho = 2{\cdot}0$.

The quantities of physical interest were also calculated numerically. Their values for various values of α are given in Table 2.

TABLE 1

VALUES OF $a(t)$ AND $b(t)$ FOR VARIOUS VALUES OF α

t	$\alpha = 0{\cdot}4$	$\alpha = 0{\cdot}5$	$\alpha = 0{\cdot}6$	$\alpha = 0{\cdot}7$	$\alpha = 0{\cdot}8$	$\alpha = 0{\cdot}9$
			$a(t)$			
0·0	1·610502	0·945766	0·851306	0·729272	0·646074	0·874525
0·1	—2·030301	2·215590	2·424081	2·523310	2·483664	2·616739
0·2	2·377681	—2·105161	—2·380990	—2·469027	—2·361796	—2·110441
0·3	2·099760	—1·197911	—1·329901	—1·418553	—1·374070	—1·467455
0·4	1·725797	—0·302368	—0·318659	—0·341436	—0·370513	—0·487782
0·5	1·504941	0·284561	+0·317193	0·321143	0·298713	0·195983
0·6	1·363574	0·767976	0·709906	0·776887	0·683910	0·635821
0·7	—13·245369	0·943963	0·998520	0·890971	0·915634	0·929253
0·8	1·133612	—1·164640	—1·175746	—1·139291	—1·069020	—1·144296
0·9	1·012538	—1·360071	—1·357158	—1·273169	—1·191533	—1·319490
1·0	0·879761	1·546469	1·835426	1·667255	1·295728	1·476437
			$b(t)$			
0·0	1·610249	0·944868	0·847788	0·729271	0·646092	0·874525
0·1	5·507357	—4·789946	—1·872559	—4·404738	—3·580390	—2·871389
0·2	3·088808	—1·382808	0·092798	1·208999	1·733528	1·994724
0·3	—0·105358	1·324798	2·139254	2·316567	2·037190	1·519741
0·4	—1·527387	1·939778	1·881728	1·515115	1·145640	0·835089
0·5	—1·569660	1·611845	1·233096	0·622642	0·266051	0·127000
0·6	—0·928177	1·037781	0·488903	0·034279	—0·232163	—0·230633
0·7	—0·046548	0·179700	—0·050209	—0·230357	—0·467243	—0·331967
0·8	—1·804925	0·064093	—0·379267	—0·551343	—0·522895	—0·306880
0·9	1·640006	—0·239908	—0·561321	—0·591715	—0·464479	—0·199006
1·0	1·460349	0·430839	0·630592	0·572038	0·337459	0·083640

TABLE 2

VALUES OF STRESS INTENSITY FACTOR, CRACK ENERGY AND DISPLACEMENT VECTOR FOR VARIOUS VALUES OF α

α	$\sqrt{2K}/p_o$	$\sqrt{2K'}/p_o$	$EW/2\pi(1-\eta^2)p_o^2$	$\frac{Eu_\theta(r,\theta)}{2(1-\eta^2)p_o}$, $r=0$: $\theta = 0$	$\theta = \pi/2$
0·4	1·550020	0·879761	3·916553	2·515766	2·812636
0·5	0·301587	1·546469	1·419600	0·216593	0·671944
0·6	0·485558	1·835246	1·723601	0·426968	0·769388
0·7	0·474792	1·667255	1·063726	0·399072	0·930902
0·8	0·300339	1·295728	0·900061	0·325612	1·014110
0·9	0·078562	1·476437	1·222053	0·914185	0·976187

APPENDIX

The results presented below can be found in[9]

$$(i)\quad \int_0^\infty e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \cos\theta) \\ \sin(\xi r \cos\theta) \end{Bmatrix} d\xi = \sum_{n=0}^\infty \frac{(\frac{1}{2})_n}{\lfloor n} \frac{t^{2n}}{r^{2n+1}} \begin{Bmatrix} \sin(2n+1)\theta \\ \cos(2n+1)\theta \end{Bmatrix}$$

$$(ii)\quad \int_0^\infty e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \sin\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi = \sum_{n=0}^\infty (-1)^n \frac{(\frac{1}{2})_n}{\lfloor n} \frac{t^{2n}}{r^{2n+1}} \begin{Bmatrix} \cos(2n+1)\theta \\ \sin(2n+1)\theta \end{Bmatrix}$$

with the help of above integrals it is quite simple to derive the following results :

$$(iii)\quad \int_0^\infty \xi e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \cos\theta) \\ \sin(\xi r \cos\theta) \end{Bmatrix} d\xi = \mp \sum_{n=0}^\infty \frac{(\frac{1}{2})_n}{\lfloor n} (2n+1) \frac{t^{2n}}{r^{2n+2}} \begin{Bmatrix} \cos(2n+1)\theta \\ \sin(2n+1)\theta \end{Bmatrix}$$

$$(iv)\quad \int_0^\infty \xi^2 e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \cos\theta) \\ \sin(\xi r \cos\theta) \end{Bmatrix} d\xi = -\sum_{n=0}^\infty \frac{(\frac{1}{2})_n}{\lfloor n} (2n+1)(2n+2) \frac{t^{2n}}{r^{2n+3}} \begin{Bmatrix} \sin(2n+3)\theta \\ \cos(2n+3)\theta \end{Bmatrix}$$

$$(v)\quad \int_0^\infty \xi e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \sin\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi = \sum_{n=0}^\infty (-1)^n \frac{(\frac{1}{2})_n}{\lfloor n} (2n+1) \frac{t^{2n}}{r^{2n+3}} \begin{Bmatrix} \cos(2n+2)\theta \\ \sin(2n+2)\theta \end{Bmatrix}$$

$$(vi)\quad \int_0^\infty \xi^2 e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \sin\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi = \sum_{n=0}^\infty (-1)^n \frac{(\frac{1}{2})_n}{\lfloor n} (2n+1)(2n+2) \frac{t^{2n}}{r^{2n+3}} \begin{Bmatrix} \cos(2n+3)\theta \\ \sin(2n+3)\theta \end{Bmatrix}$$

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