INTERNAL BALLISTICS OF HIGH VELOCITY SPECIAL PURPOSE GUNS

V. K. Gupta

Directorate of Vehicles, New Delhi

(*Received* 20 *January* 1975; *revised* 20 *October* 1975)

More and more conventional guns are being utilized as special purpose guns to achieve very high velocities by using unconventionally high C/W ratios. The existing methods of internal ballistics give satisfactory results only for low (less than one) C/W ratios. In the present paper the basic internal ballistic equations have been modified to cater for non-linear rate of burning, cubical form function and a realistic pressure gradient between breech face and the projectile base. The equations have been numerically solved. The results for low and high C/W ratios have been compared with those obtained by using conventional methods.

The use of guns for firing projectiles at high speeds has gained a special importance with the opening of space age. Simulation of high speed environments to aid research work on the aero-dynamic performance of various missile configuration, experiments on the machining of metals at ultra high speeds and launching of probes for upper atmosphere and space research are some of the important uses of high velocity guns. The problem of Gun launched space probes has been discussed by Cox. It has been stated that Dr. G. V. Bull of the Canadian Armament Research and Development considered the possibility of using Gun launched probes in 1958. It has been found[1,2] that the use of modified standard ordnance represents the most economic solution. Muzzle velocities as high as 5200 and 5800 ft/sec. have been realised at US Army Ballistics Research Laboratories by firing fin-stabilized projectiles from 5-in. and 7-in. guns smooth-bored by machining away the rifling from normal service barrels[3]. It has been stated that the quantity of propellant charge was about three times the projectile weight. A C/W ratio of about six was used to realise muzzle velocities between 8000 and 9000 ft/sec at Ballistic Research Laboratories, Aberdeen Proving Ground, U.S.A., with a 0·60 smooth bore gun[4]. Muzzle velocities of the order of 8000 ft/sec are said to be realised by firing from 90 mm smooth bore guns[5].

One important factor to be observed from above is that unconventionally high charge weight/projectile weight ratios have to be employed to realise higher velocities of this order. In the existing methods of internal ballistics it has been made very clear that the ratio C/W should not be more than for conventional guns to get realistic results[6], which are usually less than one. Arndt and McHenry,[2] presented a modification of the Hunts Hinds tabular method based on some empherical relations for such larger. C/W ratios. Murphy, Badhwar and Lavoie[7] have discussed an Interior Ballistics Calculation system for conventional and light gas guns, based on the numerical integration by finite difference method, of the simultaneous partial differential equations describing the gas to be retained in their general form.

In the following the conventional internal ballistic equations have been modified to incorporate the realistic value of pressure gradient in the Lagranges ballistics problem and a simple method is presented for the solution of these equations numerically. A cubic form function and non-linear rate of burning has been assumed. The method has been employed to calculate the ballistics and pressure distribution along the barrel for different guns using conventional and higher charge weight/projectile weight ratios. It has been shown that though for low C/W values the classical method gives quite realistic results, they fail badly for high C/W ratios.

NOTATIONS

A=Gun bore area

C=Mass of projectile

D=Propellant web size

f = Fraction of 'D' remaining at time 't'

$l = (U-C/\delta)/A$

P_b=Breech pressure at time 't'

P_m=Mean gas pressure at time 't'

P_s=Projectile base pressure at time 't'

R=Propellant gas constant

T_0=Propellant adiabatic flame temperature

T=Gas temperature at time 't'

t=Time from shot-start

U=Gun chamber capacity

V=Volume included between the breech face and base of the shot

v=Shot velocity at time 't'

W=Projectile mass corrected for spin etc.

x=Shot-travel at time 't'

z=Fraction of charge mass burnt at time 't'

β=Propellant rate of burning constant

α=Pressure index for non-linear rate of burning

$\bar{\gamma}$=Propellant gas specific heat ratio corrected for heat energy loss through the barrel

η=Propellant gas co-volume

δ=Propellant density

Subscripts

0=Referes to shot-start

1=Refers to maximum pressure

2=Refers to 'all-burnt'

3=Refers to muzzle end

THEORY

The four basic internal ballistic equations describe (*i*) burning of the propellant, (*ii*) form function *i.e.*, equation of the propellant shape, (*iii*) equation of motion of the projectile and (*iv*) gas expansion i.e., Resal equation.

Burning of Propellant—In the classical method by Hunt & Hinds[6], the rate of burning is assumed to be proportional to the space mean pressure which implies that at any time the solid portion of the unburnt propellant is uniformally distributed throughout the bore at the back of the projectile. It has, however, been shown[6] that the motion of the propellant charge during burning is negligible until burning is very nearly complete. This fact has been confirmed by Goode[8] and later by Murphy, Badhwar and Lavoie[7] wherein it has been shown that the results of ~~calculations for~~ stationary propellant agree much better with experimental results than do the results for moving propellant. As such in the present theory the rate of burning has been assumed to be proportional to the breech pressure i.e.,

$$\frac{df}{dt} = -\frac{\beta}{D} P_b^{\alpha} \tag{1}$$

The following cubical form function has been assumed to accommodate general shapes including that of spherical powders

$$z = (1-f)\,(a + bf + cf^2) \tag{2}$$

Equation of the motion of the projectile

$$W\frac{dv}{dt} = A P_s \tag{3}$$

and the Resal's equation

$$P_m = \frac{C z R T_0 - (\bar{\gamma}-1)\int_0^x A P_s\, dx}{U + Ax - \{C(1-z)/\delta\} - Cz\eta} \tag{4}$$

Pressure Gradient—The pressure gradient from the breech to the base of the projectile is mainly due to two causes, namely the inertia of the propellant gases and the gas frictional forces at the bore surfaces. The classical expressions for the pressure gradient are

$$P_s = \frac{P_m}{1 + C/3W} = \frac{P_b}{1 + C/2W} \tag{5}$$

In deducing these relations it has been assumed that the propellant is all burnt before the projectile starts to move. It has been shown by Thornhill that these relations hold good also if the unburnt propellant at any time is uniformly distributed throughout the bore which is possible only if the mean velocity of the unburnt propellant at a time is equal to the mean gas velocity i.e. $v/2$. If it was true the burning of the propellant should have taken place in the mean pressure and the form of classical pressure gradient would have been true at least for the inertia pressure gradient. But in view of the fact that the propellant stays in the chamber until last stages of burning the equation (5) are quite unrealistic except for low C/W ratios, the point of demarcation found to be[2] 0·8. It may be noted that Goldie and Coppock[10] who took the rate of burning proportional to the pressure at the breech also accepted the classical pressure gradient.

Putting $\rho = Cz/V$, Thornhill[9] obtained the following expression for the inertia pressure gradient:

$$\frac{P - P_s}{P_s} = \frac{(1 - \sigma^2)}{2\,W}\, C\, \frac{d}{dv}\,(z\,v) \tag{6}$$

$$P_m = \frac{1}{3}\,(2\,P_b + P_s) \tag{7}$$

At breech $\sigma = 0$, therefore

$$P_b = P_s \left\{ 1 + \frac{C}{2\,W}\,\frac{d}{dv}\,(z\,v) \right\} \tag{8}$$

and

$$P_m = P_s \left\{ 1 + \frac{C}{3\,W}\,\frac{d}{dv}\,(z\,v) \right\} \tag{9}$$

A good number of papers have been published by Chug[11], Prasad[12] Aggarwal and Varma[13] on the analytical solution of the problem based on this and modified density of propellant gases.

Changing the independant variable in equations (8) ~~and (9) from~~ 'v' to 'f' using (1) and (3). These can be transformed into

$$P_s = P_b + \frac{C}{2A} \cdot \frac{\beta}{D}\, P_b^{a} \left(z'\,v + v'\,z \right) \tag{10}$$

$$P_m = P_b + \frac{C}{6A} \cdot \frac{\beta}{D}\, P_b^{a} \left(z'\,v + v'\,z \right) \tag{11}$$

where

$$z' = \frac{dz}{df} = (b - a) + 2\,f\,(c - b) - 3\,c\,f^2 \tag{12}$$

and

$$v' = \frac{dv}{df}$$

similarly changing the independant variable in (3) from 't' to 'f' using (1)

$$\frac{dv}{df} = v' = - \frac{A}{W} \cdot \frac{D\,P_s}{\beta\,P_b^{a}}$$

using eq. (10)

$$v' = - \frac{A}{W} \cdot \frac{D}{\beta\,P_b^{a}} \left[P_b + \frac{C}{2A} \cdot \frac{\beta}{D}\, P_b^{a}\,(z'\,v + v'\,z) \right]$$

or

$$v' = - \frac{\left[\dfrac{A\,D}{\beta}\, P_b^{1-a} + \dfrac{C\,z'\,v}{2} \right]}{W + C\,z/2} \tag{13}$$

Kinetic energy term in Resal's equation—Following Thornhill[9] the value of $\int_0^x P_s D X$ in eq. (4) may be written as equal to

$$\text{K.E.} = \tfrac{1}{2} \left(W + \frac{1}{3}\, C\,z \right) v^2 \tag{14}$$

and the mean gas temperature at any time is given by

$$T = T_0 \left[1 - \frac{1}{2}\,(\gamma - 1) \left(W + \frac{C\,z}{3} \right) v^2 \right] \Big/ \, C\,z\,R\,T_0 \tag{15}$$

INTERNAL BALLISTIC EQUATIONS

In the ballistic solution for conventional guns, there are three possible conditions :

(i) Before the projectile starts to move—the condition is that of closed vessel and the equations are

$$P_b = P_m = P_s \tag{16}$$

$$P_b = \frac{C\,z\,R\,T_0}{A\,l - (\eta - 1/\delta)\,C\,z} \tag{17}$$

and equation (2)

(ii) From shot-start to 'all-burnt' or shot-exit whichever is earlier, the equation may be summ arised as

$$\frac{dt}{df} = -\frac{D}{\beta}\,P_b^{-\alpha}$$

$$\frac{dx}{df} = -\frac{v\,D}{\beta}\,P_b^{-\alpha} \tag{18}$$

$$\frac{dv}{df} = v' \tag{19}$$

$$v' = -\frac{\left[\frac{A\,D}{\beta}\,P_b^{1-\alpha} + \frac{Czv}{2}\right]}{W + C\,z/2} \tag{20}$$

$$z = (1-f)\,(a + bf + cf^2)$$

$$z' = (b-a) + 2f\,(c-b) - 3\,c f^2$$

$$P_b = \frac{C\,z\,R\,T_0 - \frac{\overline{\gamma}-1}{2}\,(W + C\,z/3)\,v^2}{\left[A\,(x+l) - C\,z\left(\eta - \frac{1}{\delta}\right)\right]\left[1 + \frac{C}{6\,A}\cdot\frac{\beta}{D}\,P_b^{\alpha-1}\left(z'\,v + z\,v'\right)\right]} \tag{21}$$

$$P_s = P_b + \frac{C}{2\,A}\cdot\frac{\beta}{D}\,P_b^{\alpha}\left(z'\,v + v'\,z\right)$$

$$P_m = P_b + \frac{C}{6\,A} + \frac{\beta}{D}\,P_b^{\alpha}\left[z'\,v + v'\,z\right]$$

Equations (18) and (19) have been realised by changing the independant variable from 't' to 'f' and writing $v = dx/dt$. Equation (21) has been written from (4) by replacing P_m for P_b using (11) and writing kinetic energy term from (14)

(iii) From 'all-burnt' to shot-exit, if the propellant is all-burnt before the muzzle. The equation[14] may be written as

$$\frac{dv}{dx} = \frac{A\,P_s}{v} \tag{22}$$

$$P_b = P_{b2}\left[\frac{x_2 + l'}{x + l}\right]^{\gamma} \tag{23}$$

$$\frac{dt}{dx} = \frac{1}{v} \tag{24}$$

and

$$P_s = \frac{P_m}{1 + C/3\,W} = \frac{P_b}{1 + C/2W}$$

METHOD OF SOLUTION

The initial conditions are, $x, v, z, t, P_b, P_m, P_s$, are equal to zero and 'f' $= 1$. 'f' is decreased in steps '$d\,f$' to calculate 'z' from (2) which gives P_b from (17). This value of P_b is compared with P_o the shot-start pressure. If P_b exceeds P_0 the value of 'f_0' is interpolated for $P_b = P_0$, it gives the conditions at shot-start. The following 'defined functions' may be used for the solution :

$$\frac{dt}{df} = \psi_1 (P_b)$$

$$P_b = \psi_2 (z, v, x, z', v', P_b) \qquad v' = \psi_5 (P_b, z', v, z')$$

$$z = \psi_3 (f) \qquad P_s = \psi_6 (P_b, z', v, v')$$

$$z' = \psi_4 (f) \qquad P_m = \psi_7 (P_b, z', v, v')$$

$$\frac{dv}{df} = v' \qquad \frac{dx}{df} = \psi_8 (P_0, v)$$

$\psi_1\ \psi_2\ \ldots\ldots\ldots\ldots\ \psi_8$ are the functions of variable as shown in the parenthesis. Initial conditions are $x = 0$, $v = 0$, $t = 0$, $z' = \psi_5 (f_0)$, $v' = \psi_6 (P_{b0}, z'_0, 0, z_0)$. The independent variable 'f' is decreased in steps of 'df' and the equations are solved by Runge-Kutta method of solution of simultaneous equations. The method used is similar to one used by Gupta [14,15]. From above, it may be noticed that equation (21) is nonlinear in P_b if $a \neq 1$. During the step by step solution the value of P_b' in the previous step is always available in the computer memory. To calculate P_b, the equation is solved by iteraction. The consistancy is achieved in only 2 to 3 steps. After the 'all-burnt' the solution is just similar to the method given by Gupta[15].

APPLICATIONS

The principal problem of theoretical internal ballistics is to predict muzzle velocity, maximum pressure and its distribution along the barrel and the 'all-burnt' position, with the given gun, projectile and propellant characteristics. It is expected that a good method besides its simplicity and less time consuming should give the better prediction of these parameters. The method discussed in the previous section has been applied to calculate the ballistics for the following propellant charges :

(*i*) 3·7-in. AA Gun with characteristics[8]

(*ii*) 6-in. Naval Gun with 31 lb and 44 lb charge weights[8]

(*iii*) 40 mm smooth bore proposed gun[2].

In the first two cases the maximum C/W ratio is only 0·44 while for 40 mm gun C/W from 0·25 to 6 has been considered. For the sake of comparison internal ballistics parameters have been calculated under the following four assumption:

Method 'A'— Rate of burning proportional to breech pressure with modified pressure gradient (Method proposed in this paper)

Method 'B'— Rate of burning proportional to breech pressure with classical pressure gradient (Assumptions of Goldie & Coppock).

Method 'C' — Rate of burning proportional to mean pressure with classical pressure gradient (Hunt & Hinds).

Method 'D'— Rate of burning proportional to mean pressure with modified pressure gradient.

The assumptions of 'D' are obviously contradictory. As discussed earlier in this paper if the propellant is moving with the shot and is uniformly distributed through the bore, conventional pressure gradient holds good and if the propellant stays in, the chamber rate of burning should be proportional to the breech pressure. The equations solved numerically by method 'B', 'C' and 'D' have been summarized in Appendix 'A'.

RESULTS AND DISCUSSION

In case of 3.7-in. and 6-in. guns, there is not much of difference in the ballistic curves shown in Fig. 1 and 2. It may, however, be noticed that different values of shot-start pressure, have to be taken in order to match the experimental maximum pressure. Shot-start pressure takes into account, not only the effect of resistance to initial motion of the projectile but also the subsequent bore resistance, errors arising out on account of use of closed vessel rate of burning constants, assumptions regarding pressure for rate of burning and the pressure gredient. In fact Cornor[16] has warned that not much physical significance can be attached to the shot-start pressure.

The same value of shot-start pressure i.e. 6 tsi has been assumed with method 'A' and 'C' for the 40 mm gun. This gun is taken to be a model in which by some mechanical means the shot is not allowed to move till

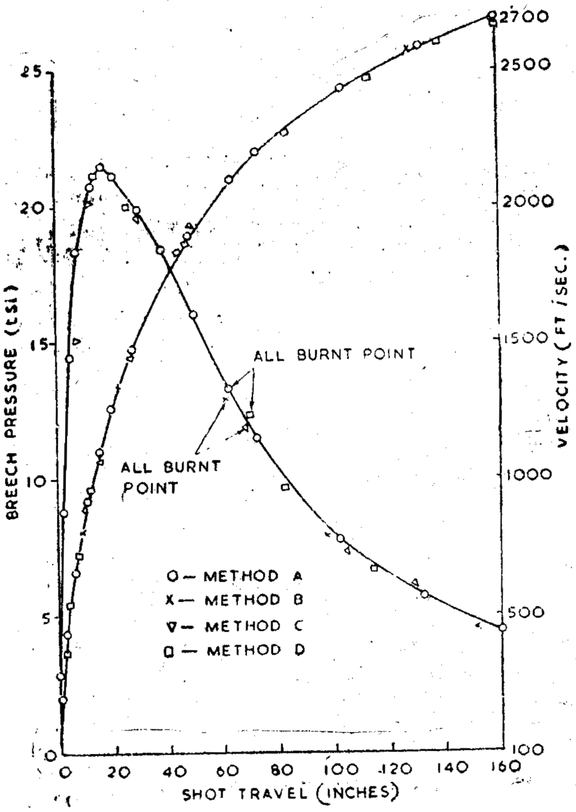

Fig. 1— Ballistics curves calculated under different assumptions for 3·7-in. AA Gun.

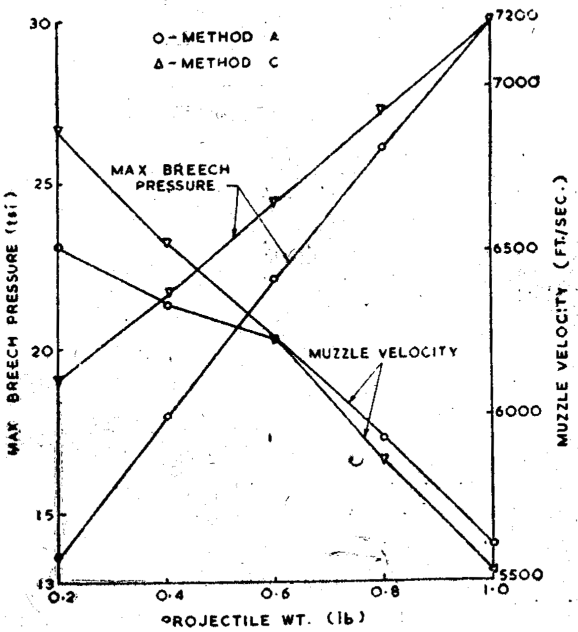

Fig. 3— Maximum breech pressure, muzzle velocity for 40 mm Gun having charge weight 1·0 lb.

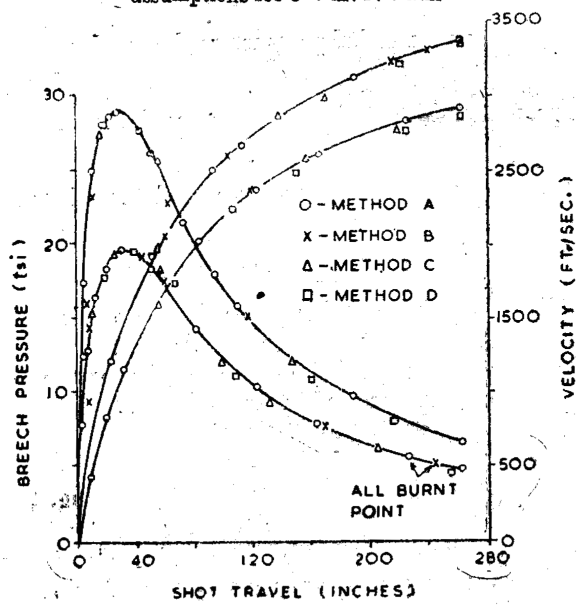

Fig. 2— Ballistics curves calculated under different assumptions for 6-in. Gun with 31 lb and 44 lb charge weights.

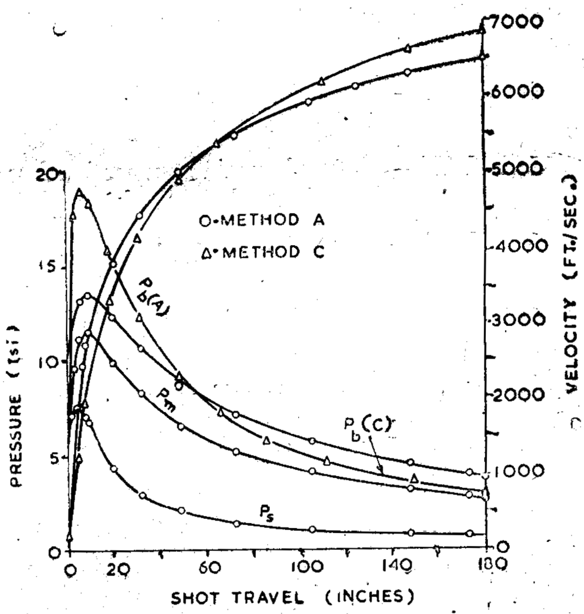

Fig. 4— The oretical pressure, velocity and space curve for 40 mm Gun, projectile weight 0·2 lb, charge weight 1·0 lb.

6 tsi pressure is generated in the chamber and does not come across any bore resistance later. Due to the different erosive burning characteristics, which is a factor of velocity of gases passing over the unburnt propellant, with different C/W ratio the shot-start pressure may not be the same to match the experimental pressure. As such it is doubtful if the shot-start pressure can be used with much confidence as a data to realise higher muzzle velocity[2]. Corner[16], while calculating the effect on muzzle velocity by bore resistance has, however shown that resistance occurring sufficiently early in the shot-travel, the muzzle velocity rises while if the resistance occurs further down the bore the muzzle velocity falls.

With the increase in C/W ratio the difference in maximum pressure calculated by the two methods, as shown in Fig. 3, increases[1,9,3]. With C/W-6 the difference being 9·3 tsi as compared to zero for C/W-1·0. There is a significant difference in the muzzle velocity calculated by the two methods especially with large C/W ratios. The muzzle velocities as calculated by Arndt & McHenry[2] appears to be on a very high side, however, without experimental evidence nothing with confidence can be said. In Fig. 4 breech, mean and base pressures calculated by the new method have been plotted against shot-travel. The three pressures do not attain their maximum value at the same point as in the case with conventional pressure gradient. The maximum base

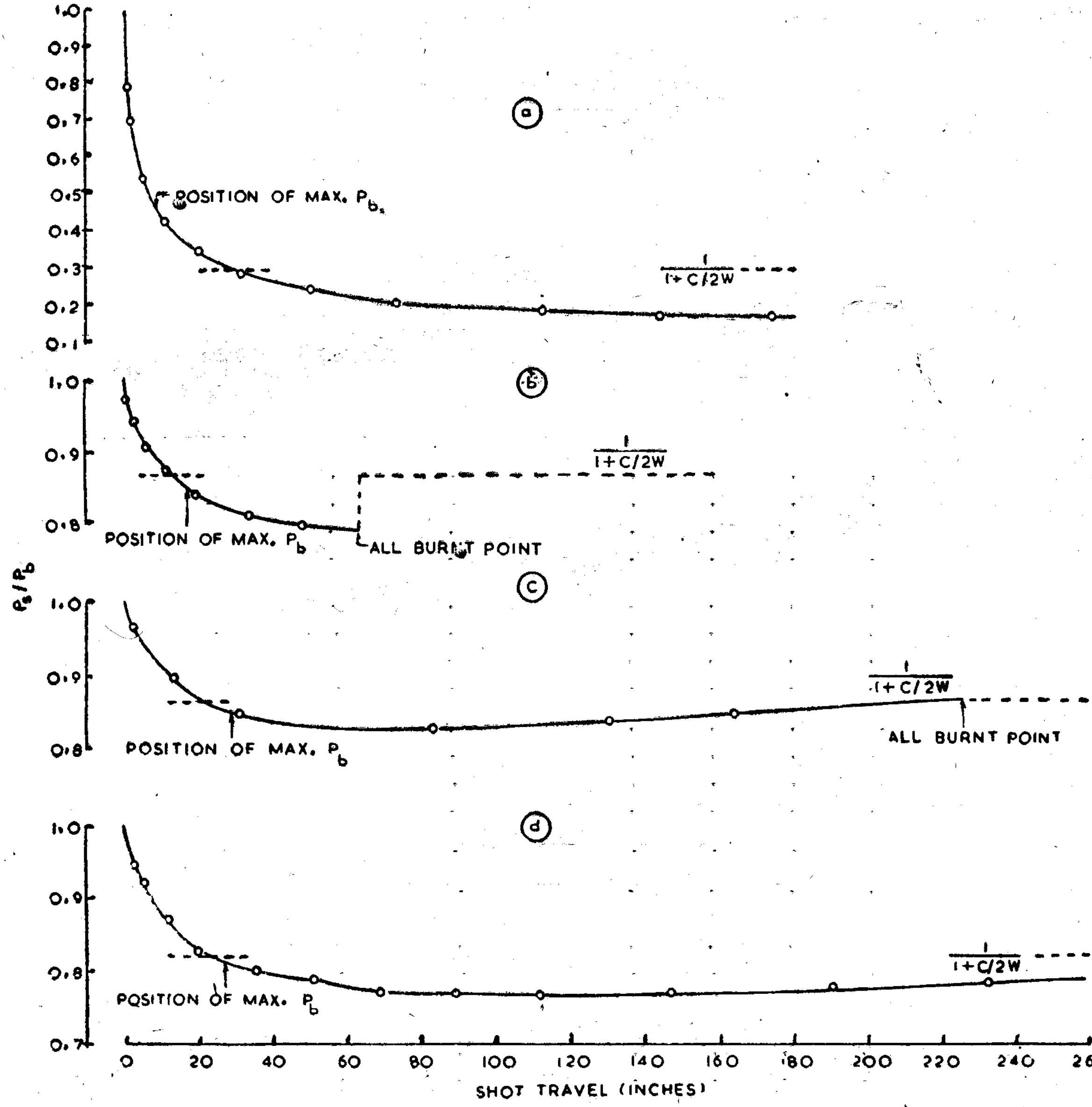

Fig. 5—Theoretical pressure gradient for (a) 40 mm Gun, C=1.0 lb, W=0.2 lb; (b) 3.7-in.AA Gun, C=8 lb 50 oz, W=28 lb ; (c) 6-in. Gun, C=31 lb, W=100 lb and (d) 6-in. Gun, C=44 lb, W=100 lb.

pressure occurs earliest. It is interesting to note that same conclusion has been arrived at by Murphy, Badhwar and Lavoie[7] who calculated pressure space curves for 3·2-in. smooth bore gun with a C/W ratio of 5.

The ratio P_s/P_b has been plotted against shot-travel in Fig. 5. The curves for 6-in. Naval Gun may be scrutinized more carefully for their similarity with the curves given in HMSO[5]. It may also be noticed that ratio P_s/P_b is more than $1 + C/2W$ at first and less than $1+C/2W$ towards the end of shot-travel and further the time at which P_s/P_b-$1/(1+C/2W)$ is little before peak pressure for conventional guns. Similar findings have been reported by Corner[16].

Discontinuity at 'all-burnt' point—On account of completely neglecting the motion of solid portion of the propellant, a discontinuity is noticed at 'all-burnt' point for 3·7-in. gun.

Maximum possible muzzle velocity—This matter is of considerable importance especially for this paper. Experience shows that the projectile velocity cannot be increased arbitrarily by increasing the propellant charge. The reason for this is already mentioned pressure difference between breech block and the projectile base to which a great part of the energy is transformed into the energy of movement of the propellant gases. From *Table 1*, it may be noticed that with C/W ratio of six, the maximum shot base pressure is only 8·8 its as against 20·4 tsi breech pressure. Corner[16] gave the maximum velocity for an evacuated barrel to be:

$$V_m = \frac{2}{\gamma - 1} \left(\gamma R T_0\right)^{1/2}$$

Experiments conducted with nitro-cellulose powder on a weapon of calibre 7·9 mm by Land Weider[17] yielded about 9000 ft/sec. as the highest velocity (C=11 grams, $W = 0·25$ grams).

TABLE 1

THEORETICAL BALLISTIC PARAMETERS CALCULATED BY FOUR METHODS

Equipment	Method	Shot-start press	Maximum pressure (tsi) P_b	P_m	P_s	Muzzle velocity (ft/sec.)	position of Max. P_b (in.)	All- burnt point (in.)	As fired ballistics	Remarks
3·7-in. AA	A	2·7	21·4	20·4	18·5	2710	17·0	63·5	MV=2701	With method A and
	B	1·7	21·4	20·5	18·6	2707	16·7	63·6	P_b = 21·4	D max. P_b, P_m, P_s
	C	2·7	21·4	20·5	18·6	2703	15·1	70·3		did not occur at the
	D	3·3	21·4	20·4	18·6	2688	15·7	71·1		same position.
6-in. Naval C=44 lbs.	A	2·2	29·0	27·2	23·8	3338	27·1	Out (98·0)	MV=3356	In the parenthesis for
	B	0·1	29·0	27·1	23·6	3359	28·6	Out (98·0)	P_b =29·0	'all-burnt' point, of
	C	2·0	29·0	27·7	23·6	3341	24·3	Out (97·8)		shows % quantity of
	D	3·4	29·0	27·3	24·1	3315	25·3	Out (96·2)		charge consumed.
6-in. Naval C=31 lbs.	A	0·95	19·6	18·6	16·6	2874	30·0	234	MV=2850	
	B	0·1	19·6	18·7	17·0	2880	29·0	242	P_b=19·6	
	C	0·95	19·6	18·7	17·0	2876	28·0	262		
	D	1·5	19·6	18·6	16·8	2862	29·0	262		
40 mm C=1·2 lb	A	6·0	20·4	15·8	8·8	7190	11·8	142	Not available	
W=0·2 lb	C	6·0	29·7	22·3	7·5	7923	7·2	Out (92·1)		
40 mm C=1·0 lb	A	6·0	13·6	11·2	7·4	6508	8·6	Out (79·8)	Not available	The calculated values in Ref. 2, P. 266. MV ~7000 ft/s. P_b ~ 17·0 tsi.
W=0·2 lb	C	6·0	19·0	14·5	5·5	6864	5·8	Out (75·9)		
40 mm C=0·2 lb	A	6·0	6·5	6·3	6·0	2851	0·8	Out (99·2)	Not available	
W=0·8 lb	C	6·0	6·5	6·3	6·0	2841	0·8	Out (97·7)		

REFERENCES

1. COX, R. N., *New Scientist*, **36** (1967), 337.
2. ARNDT, G. & MCHENRY, J. T., *Explosivstoffe*, **18** (1970), 253.
3. MURPHY, C. H. & BULL, G. V., 'Review of the High Altitude Research Program', (The Fluid Dynamic Aspects of ballistics, AGARD Proceedings No. 10), Sept. 1966, p. 407.
4. ARMENDT, BRADSHAW, F. & CREWS GEORGE, C., *American Refleman*, **110** (1962), 17.
5. FINGER, DENIE . W., 'Telemetering Projectile for High Velocity Guns, Advances in Hyper-velocity Techniques', (Proceeding of the Second Symposium on Hyper-velocity Techniques, Plenum Press, New York), 1962, p. 211.
6. Internal Ballistics, HMSO, 1951, p. 78 & 80.
7. MURPHY, J. R. B., BADHWAR, L. K. & LAVOIE, G. A., 'Interior Ballistics Calculation System for Light Gas Guns and Conventional Guns', (The Fluid Dynamic Aspects of Ballistics, NATO. AGARD Conference Proceedings No. 10), Sept. 1966, p. 51.
8. GOODE, J. B., 'Forty Years of British Internal Ballistic Research', (Selected Topics on Ballistics, AGARDograph No. 32, Permgamon), 1959, p. 216.
9. Thornhill., Internal Ballistics HMSO, 1951, p. 76.
10. GOLDIE & COPPOCK., 'Theory of the Interior Ballistics of Guns, (John Wiley, New York), 1950, p. 174.
11. CHUGH, O. P., *Explosivstoffe*, **15** (1967), 250.
12. PRASAD, G. R., *Explosivstoffe*, **18** (1970), 66.
13. AGGARWAL, S. P. & VERMA, P. S., *Memorial de L'Artillerie, Francaise*, **2** (1973), 615-530.
14. GUPTA, V. K., *Def. Sci. J.*, **25** (1975), 127.
15. GUPTA, V. K., *Def. Sci. J.*, **25** (1975), 85.
16. CORNER, J., 'Theory of the Interior Ballistics of Guns' (John Wiley, New York), 1950, p. 191, 220 & 357.
17. LAND WEIDER., 'BALLISTIK', Part II, Edited by KUTTERER, Ing. Richard Emil, 1912, p. 41.

APPEDIX 'A'

INTERNAL BALLISTIC EQUATIONS UNDER OTHER ASSUMPTIONS IN RESPECT OF RATE OF BURNING AND PRESSURE GRADIENT

(*i*) Rate of burning proportional to breech pressure and conventional pressure gradient—Method 'B'

$$\text{I} \qquad \frac{dt}{df} = -\frac{D}{\beta} P_b^{-\alpha}$$

$$\text{II} \qquad z = (1-f)\ (a + bf + cf^2)$$

$$\text{III} \qquad \frac{dv}{df} = -\frac{A\,D\,P_c^{1-\alpha}}{\beta\,W\left(1+\dfrac{C}{2\,W}\right)}$$

$$\text{IV} \qquad P_b = \frac{Cz\ RT_0 - \frac{1}{2}\ (\bar{\gamma}-1)\ (W + C/3)\,v^2}{U + Ax - C\ (1-z)/\delta - Cz\eta}\left[\frac{1+\dfrac{C}{2\,W}}{1+\dfrac{C}{3\,W}}\right]$$

$$\text{V} \qquad P_s = \frac{P_m}{1 + C/3\,W} = \frac{P_b}{1 + C/2\,W}$$

(*ii*) Rate of burning proportional to mean pressure and conventional pressure gradient—Method 'C'

$$\text{VI} \qquad \frac{dt}{df} = -\frac{D}{\beta} P_m^{-\alpha}$$

$$\text{VII} \qquad \frac{dv}{df} = -\frac{AD\,P_m^{1-\alpha}}{\beta\,W\left(1+\dfrac{C}{3\,W}\right)}$$

$$\text{VIII} \qquad P_m = \frac{Cz\,RT_0 - \frac{1}{2}\ (\bar{\gamma}-1),\ (W + C/3)\ v^2}{U + Ax - C\ (1-z)/\delta - Cz\eta}$$ and eqs. II & V

(*iii*) Rate of burning proportional to mean pressure and modified pressure gradient—Method 'D'

IX $\quad z^1 = (b - a) - 2f\ (b - c) - 3\,c f^2$

X $$\frac{dv}{df} = - \frac{\left[\dfrac{A\,D}{\beta}\ P_m^{1-a} + \dfrac{C\,z'\,v}{3}\right]}{\left(W + \dfrac{C\,z}{3}\right)}$$

XI $$P_m = \frac{C\,z\,R\,T_\circ - \dfrac{\bar{\gamma} - 1}{2}\left(W + \dfrac{C\,z}{3}\right) v^2}{A\ (x + l) - C\,z\left(\eta - \dfrac{1}{\delta}\right)}$$

XII $$P_b = P_m - \frac{C\,\beta}{6\,A\,D}\ P_m^{a}\,(z'\,v + v'\,z)$$

XIII $$P_s = P_m + \frac{C\,\beta}{3\,A\,D}\ P_m^{a}\,(z'\,v + v'\,z)$$ and eqs. II & VI