LAMINAR THREE-DIMENSIONAL BOUNDARY LAYERS WITH SWIRL IN SLIP FLOW REGIME

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The effect of slip on swirling, low speed flows of slightly rarefied gases in the slip flow regime in a three dimensional axisymmetric laminar boundary layer is investigated by similarity solution approach and by use of perturbation techniques. The governing ordinary differential equations are solved numerically using Runge-Kutta-Gill method. The results show that the skin friction and the heat transfer are reduced due to the effect of slip.

Swirling flows have a wide variety of applications in many engineering devices. An extensive survey of this type of flows and their applications has been done recently by Lewellen[1] and Murthy[2]. Back[3] has studied the effect of swirl on the flow and heat transfer in a three-dimensional axisymmetric laminar compressible boundary layer.

In this paper, the effect of slip on the swirling flow in a slightly rarefied gas has been studied by introducing the appropriate boundary conditions. The present investigation has been motivated by its possible application in flights at high altitudes (about 48 km and above), where the effect of slip ought to be considered, for instance, swirling nozzle flow in a low-thrust rockets[4].

GOVERNING EQUATIONS

For gases in slip flow regime, *i.e.*, for slightly rarefied gases, it has been found that the velocity and the temperature jump are sufficiently small and that the solutions of the usual equations of continuity, momentum and energy subject to boundary conditions, modified suitably so as to incorporate the velocity slip and temperature jump conditions, are in good agreement with the experimental results[5].

Hence, following Back's analysis[3] and using the first order slip boundary conditions[5]

$$(u)_{\zeta=0} = \frac{2-\sigma}{\sigma}\,\bar{l}\left(\frac{\partial u}{\partial \zeta}\right)_{\zeta=0}, \quad (v)_{\zeta=0} = \frac{2-\sigma}{\sigma}\,\bar{l}\left(\frac{\partial v}{\partial \zeta}\right)_{\zeta=0}$$

$$(w)_{\zeta=0} = 0, \quad (H_t)_{\zeta=0} = H_w + \frac{2-\alpha'}{\alpha'}\;\frac{2\gamma}{\gamma+1}\;\frac{\bar{l}}{Pr}\left(\frac{\partial H}{\partial \zeta}\right)_{\zeta=0}, \tag{1}$$

instead of no-slip conditions $u = v = w = 0$ and $H_t = H_w$ at the surface $\zeta = 0$ considered by Back[3], the laminar boundary layer equations in dimensionless form governing the low-speed swirling flow of a perfect gas in the slip flow-regime with simplifying assumptions of constant specific heat, viscosity proportional to temperature and Prandtl number unity in a cooled stationary channel of variable cross-section, under similarity requirements are given by[3]

$$f''' + ff'' + (\bar{\alpha} + \bar{\beta})\,[G\,(1 - g_w) + g_w] - \bar{\beta}\, f'^2 - \bar{\alpha}\, G^2 = 0 \tag{2}$$

$$S'' + f\,S' = 0 \tag{3}$$

$$G'' + fG' = 0 \tag{4}$$

The boundary conditions are[6]

$$f\,(0) = 0, \quad f'\,(0) = \epsilon f''\,(0), \; f'\,(\infty) = 1 \tag{5}$$

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$$S(0) = \epsilon\, S'(0),\ S(\infty) = 1 \qquad (6)$$

$$G(0) = \epsilon'\, G'(0),\ G(\infty) = 1 \qquad (7)$$

where ξ, η, ζ are the longitudinal, circumferential and normal distances; u, v, w are the velocity components along ξ, η, ζ directions; H_t is the total enthalpy; H_w is the total enthalpy at the wall; σ and α' are the reflection and thermal accommodation coefficients respectively (both being close to unity); $\bar{l}$ is the mean free path of the gas; γ is the specific heats ratio; Pr is the Prandtl number; f', S and G are the dimensionless, longitudinal velocity, swirl velocity and total enthalpy difference ratio; $\bar{\alpha}$ and $\bar{\beta}$ are the swirl and longitudinal acceleration parameters respectively; g_w is the cooling parameter; ϵ and ϵ' are the dimensionless velocity slip and temperature jump parameters and prime denotes differentiation with respect to Z.

It may be remarked[5] that in slip flow regime (regime of slight rarefaction), the Knudsen number K ($= M/\sqrt{Re}$, where M is the freestream Mach number and Re is the Reynolds number) lies between the range $0{\cdot}01 < M/\sqrt{Re} < 0{\cdot}1$, $Re > 1$, whereas for continuum flow $M/\sqrt{Re} \leqslant 0{\cdot}01$. The slip effects[5] are of importance at an altitude of about 48 km for $M = 1$ and about 32 km for $M = 4$. As we are considering low-speed compressible boundary layer flow with $M \sim 1$, $Re \sim 10^4$, the slip parameters ϵ and ϵ', which are directly proportional to Knudsen number K, are small and $\epsilon = \epsilon'$ for all practical purposes[6].

The equations (2), (3) and (4) are the same as those of Back[3]. However, the boundary conditions (5), (6) and (7) are different when $\epsilon \neq 0$. Since S and G satisfy equations and boundary conditions of the same form, $S = G$. The solutions of (2), (3) and (4) under conditions (5), (6) and (7) with $\epsilon = 0$ (*i.e.*, for no-slip flows) have been obtained by Back[3] using quasilinearization technique. He found that the solutions for large $\bar{\alpha}$ and $\bar{\beta}$ even under simplifying assumptions mentioned earlier (*i.e.*, the product of viscosity and density is constant, Prandtl number is unity, etc.) are difficult to obtain.

APPLICATION OF PERTURBATION TECHNIQUE

The equations (2) and (4) can be solved easily through the application of regular perturbation technique by assuming the following series expansions[6] for f and G

$$f = f_0 + \epsilon f_1 + \ldots,\quad G = G_0 + \epsilon\, G_1 + \ldots \qquad (8)$$

The zeroth order solutions f_0 and G_0 correspond to the flow without slip ($\epsilon = 0$) whereas f_1 and G_1 represent the effect of slip ($\epsilon \neq 0$) to the first order and they satisfy the following sets of ordinary differential equations

Zeroth order (without slip) :

$$f_0''' + f_0 f_0'' + (\bar{\alpha} + \bar{\beta})\,[(1 - g_w)\, G_0 + g_w] - \bar{\beta} f_0'^2 - \bar{\alpha}\, G_0^2 = 0 \qquad (9)$$

$$G_0'' + f_0\, G_0' = 0 \qquad (10)$$

Boundary conditions

$$f_0(0) = f_0'(0) = G_0(0) = 0,\ f_0'(\infty) = G_0(\infty) = 1 \qquad (11)$$

First order (first order slip effect)

$$f_1''' + f_0 f_1'' + f_0'' f_1 + (\bar{\alpha} + \bar{\beta})\,(1 - g_w)\, G_1 - 2\,(\bar{\beta} f_0' f_1' + \bar{\alpha}\, G_0\, G_1) = 0 \qquad (12)$$

$$G_1'' + f_0\, G_1' + G_0' f_1 = 0 \qquad (13)$$

Boundary conditions

$$f_1(0) = 0,\ f_1'(0) = f_0''(0),\quad G_1(0) = G_0'(0),\ f_1'(\infty) = G_1(\infty) = 0 \qquad (14)$$

It may be remarked that equations (9) to (11) are identical with those of Back[3] and equations (12) and (13) are linear equations with variable coefficients.

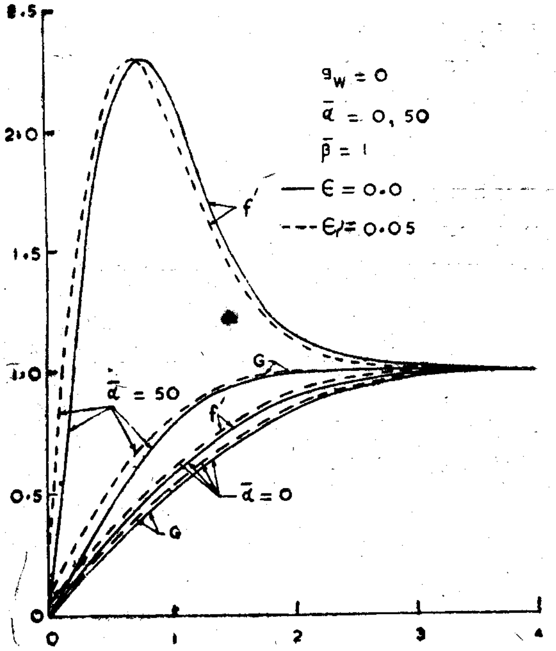

Fig. 1—Effect of slip on the longitudinal velocity and enthalpy (swirl velocity) profiles.

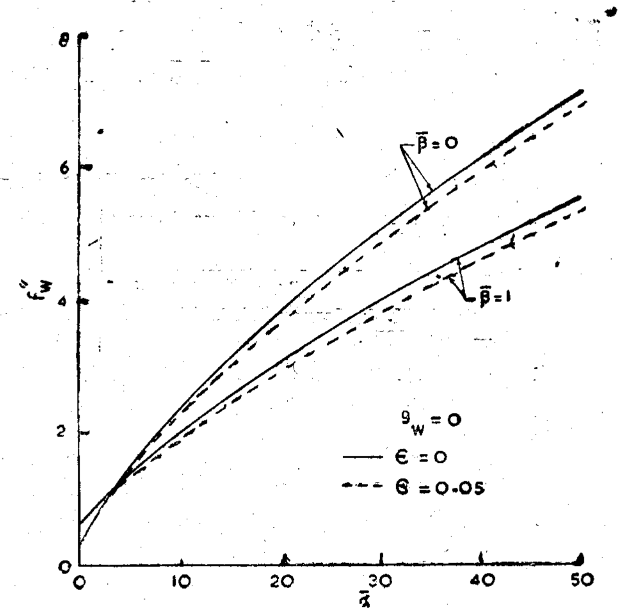

Fig. 2—Effect of slip on longitudinal friction parameter in swirling flows.

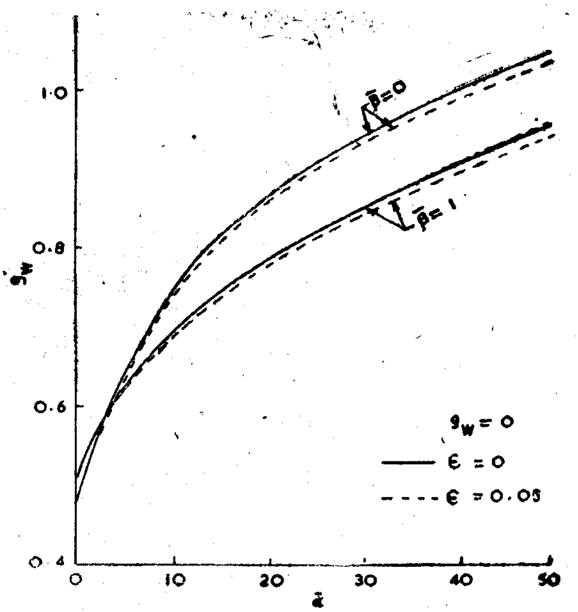

Fig. 3—Effect of slip on heat transfer parameters in swirling flows.

HEAT FLUX AND SHEAR STRESS AT THE SURFACE

The heat flux to the surface and the wall shear stresses in the longitudinal and tangential directions can be written in the form of Stanton number St and the skin friction coefficients c_{f_1} and c_{f_2} as follows[3]

$$St = \lambda G'_w/Pr_w,\ c_{f_1} = 2\lambda f_w'',\ c_{f_2} = 2(u_e/w_e)\lambda S'_w \quad (15)$$

where

$$\lambda = [1/(\rho_e u_e \bar{\xi}/\mu_e)^{1/2}]\ [C_w/2^{1/2}],\ C_w = \rho_w\ \mu_w/\rho_e\ \mu_e.$$

Here ρ and μ are density and viscosity respectively, $\bar{\xi}$ is the distance along the body[3] and the subscripts w and e denote the conditions at the wall and at the edge of the boundary layer respectively. From Back[3] we have, $S'_w = G'_w$, $Pr_w = C_w = 1$.

RESULTS AND DISCUSSION

Equations (9) and (10) under conditions (11) and equations (12) and (13) under conditions (14) have been solved numerically on a digital computer by the use of Runge-Kutta-Gill method for the various values of the parameters $\bar{\alpha}$ $(0 \leqslant \bar{\alpha} \leqslant 50)$, $\bar{\beta}$ $(\bar{\beta} = 0,1)$, g_w $(g_w = 0)$ and ϵ $(\epsilon = 0, 0{\cdot}05)$. The slip parameter $\epsilon = 0{\cdot}05$ will correspond to the height between 48 km and 60 km. The effect of slip on the longitudinal velocity profiles f' and enthalpy or swirl velocity profiles G, longitudinal skin friction parameter f''_w and heat transfer parameter G_w' are shown in Figs. 1-3. As expected, the nature of these profiles remains unaffected by the slip parameter, and velocity overshoot in the longitudinal velocity f' continues to be a characteristic feature of the flow even in the slip flow regime. The reasons for the occurrance of velocity overshoot in f' has been explained by Back[3]. The longitudinal skin friction parameter f''_w and the heat transfer (or tangential skin friction) parameter G'_w for given $\bar{\alpha}$, $\bar{\beta}$ and g_w are reduced due to the effect of the slip parameter ϵ.

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