TRANSIENT TEMPERATURE DISTRIBUTION IN AN INFINITE SLAB

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The temperature distribution in a slab of constant thickness with one face of the slab at constant temperature and the other exposed to a transient temperature field, has been obtained using the method of differential operator. Numerical values of the temperature field are found by taking the applied temperature as a continuous function of the length and time.

In a recent paper, Charles J. Martin[1] has used Fourier Transforms and theory of complex variable for the determination of temperature distribution in an infinite slab of constant finite thickness due to the application of transient temperature field. He discussed in detail the problem by taking the applied temperature as a product of two discontinuous step-functions. This paper is concerned with the determination of the temperature distribution in an infinite slab of constant finite thickness due to the application of a transient temperature field at one face of the slab. The other face of the slab is maintained at constant temperature. The solution of the heat conduction equation is obtained by 'Symbolic Method' in terms of differential operators[2], when the applied temperature is a continuous function of length and time. Numerical results at successive times are obtained by approximating the solution to a finite series. Results are presented by graphs.

THE PROBLEM AND ITS SOLUTION

Using the rectangular Cartesian system of co-ordinates x, y, z, let the infinite slab be bounded by two parallel planes $y=0$ and $y=h$ and be infinite in extent in x and z directions. The face $y=0$ is maintained at zero temperature while the face $y=h$ is exposed to a continuous surface temperature which varies with x and the time t.

The temperature θ in the slab satisfies the linear heat conduction equation[3]

$$\nabla^2\theta = \frac{\rho c}{k} \cdot \frac{\partial \theta}{\partial t}, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}$$

and k, ρ, c are the thermal conductivity, density and specific heat of the material respectively.

Since the face $y=0$ is maintained at zero temperature and $y=h$ is exposed to temperature, the boundary conditions are :

$$\theta\,(x, 0, t) = 0, \text{ for all } x \text{ and } t \tag{3}$$

$$\theta\,(x, h, t) = \theta_0 \cdot f\,(x) \cdot \phi\,(t) \tag{4}$$

where $f\,(x)$ and $\phi\,(t)$ are functions of x and t which are continuous and continuously differentiable, and

$$\left.\begin{array}{c} -\infty < x < \infty \\ 0 < y < h \end{array}\right\}$$

On substituting

$$u = \frac{x}{h}, \; v = \frac{y}{h} \tag{5}$$

$$\omega = \frac{kt}{\rho} \cdot ch^2, \; T = \frac{\theta}{\theta_0} \tag{6}$$

we have

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = \frac{\partial T}{\partial \omega}, \tag{7}$$

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and the boundary conditions are :

$$\left.\begin{aligned} T(u, 0, \omega) &= 0, \\ T(u, 1, \omega) &= f(u)\,\phi(\omega) \\ &= H(u, \omega), \text{ say} \end{aligned}\right\} \tag{8}$$

On writing $p = \dfrac{\partial}{\partial u}$ and $q = \dfrac{\partial}{\partial \omega}$, the heat conduction equation (7), on using the boundary conditions (8) yields

$$T = \frac{\sin v\sqrt{p^2 - q}}{\sin \sqrt{p^2 - q}}\, H(u, \omega) \tag{9}$$

NUMERICAL EXAMPLES

In this section, we take some particular type of applied surface temperature to illustrate our problem. Two cases are discussed.

Case—I : Taking the applied surface temperature as

$$H(u, \omega) = \sum A_n\, e^{-(a_n|u| + b_n|\omega|)}, \quad a_n\, b_n > 0 \tag{10}$$

Substituting (10) in (9) and after some calculations, we get

$$T = \sum A_n \frac{\sin v\sqrt{(a_n^2 - b_n)}}{\sin \sqrt{(a_n^2 - b_n)}}\, e^{-(a_n|u| + b_n|\omega|)} \tag{11}$$

Now by giving suitable values to the constants a_n, b_n, and A_n, it is easy to find the numerical values when the applied temperature function is any sine, cosine, hyperbolic sine or hyperbolic cosine functions of x and t.

Case—II : Taking the applied surface temperature as

$$H(u, \omega) = \frac{e^{-b|u|}}{(1 + a\omega)}, \quad \omega \geqslant 0,\ b > 1 \tag{12}$$

Putting this value in (9), we get

$$T = e^{-b|u|}\, \frac{\sin v\sqrt{b^2 - q}}{\sin \sqrt{b^2 - q}}\, \phi(\omega) \tag{13}$$

For numerical evaluation of the temperature distribution in the slab, we take $a = 0\cdot 1$ and $b = 3\pi$.

Now the expression for T given by (13) is expanded as infinite series in powers of q in the form

$$e^{-b|u|}\left[\sum_{n=0}^{\infty} \gamma_n\, q^n\right] F(\omega) \tag{14}$$

where

$$F(\omega) = \frac{1}{\sin \sqrt{b^2 - q}}\cdot \phi(\omega) \tag{15}$$

The expression for T in the form of equation (14) is

$$\begin{aligned} T = e^{-b|u|} \Bigg[& \sin bv - \frac{v\cos bv}{2b}\, q - \frac{1}{2^2}\left(\frac{v^2 \sin bv}{b^2} + \frac{v\cos bv}{b^3}\right)\frac{q^2}{2!} + \frac{1}{2^3}\left(\frac{v^3\cos bv}{b^3} - \right. \\ & \left. - \frac{3v^2 \sin bv}{b^4} - \frac{3v\cos bv}{b^5}\right)\frac{q^3}{3!} + \frac{1}{2^4}\left(\frac{v^4 \sin bv}{b^4} + \frac{6v^3\cos bv}{b^5} - \frac{15v^2 \sin bv}{b^6} - \right. \\ & \left. - \frac{15v\cos bv}{b^7}\right)\frac{q^4}{4!} + \frac{1}{2^5}\left(-\frac{v^5\cos bv}{b^5} + \frac{10v^4\cos bv}{b^6} + \frac{45v^3\cos bv}{b^7} - \frac{105v^2\sin bv}{b^8} - \right. \\ & \left. - \frac{10v\cos bv}{b^9}\right)\frac{q^5}{5!} + \ldots\ldots\ldots\ldots \Bigg] F(\omega) \end{aligned} \tag{16}$$

The coefficient γ_n in (14) are functions of v and the sequence $\{\gamma_n\}$ is seen to be rapidly convergent. The equation (15) can be written as

$$\phi(\omega) = \left(\sum_{n=0}^{\infty} \alpha_n \; q^n \right) F(\omega) \tag{17}$$

The sequence $\{\alpha_n\}$ decreases in magnitude rapidly. The first six α_n's are calculated.

$$\begin{aligned}
\alpha_0 &= 0 \cdot 000000 \\
\alpha_1 &= 0 \cdot 05305165 \\
\alpha_2 &= 0 \cdot 00014931 \\
\alpha_3 &= -\,0 \cdot 00002404 \\
\alpha_4 &= -\,10^{-6} \times 0 \cdot 24 \\
\alpha_5 &= \;\; 10^{-8} \times 0 \cdot 177
\end{aligned}$$

Now expanding the function $\phi(\omega)$ about any arbitrary point ξ by Taylor's series, we get

$$\phi(\omega) = \sum_{n=0}^{\infty} \phi_n(\xi) = \frac{(\omega - \xi)^n}{n!}, \tag{18}$$

where $\phi_n(\xi)$ is the nth derivative of $\phi(\omega)$ at $\omega = \xi$.

The terms in the equation (18) decrease rapidly in magnitude and the first four terms can be taken for approximation of $\phi(\omega)$. Thus, the function $F(\omega)$ defined by (15) is of the form

$$F(\omega) = \sum_{n=0}^{5} \beta_n (\omega - \xi)^n, \tag{19}$$

where β_n are function of ξ.

The coefficients β_n ($n = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots 5$) are determined from the following relations

$$\phi(\xi) = \sum_{n=0}^{5} n! \, \alpha_n \beta_n,$$

$$\phi_1(\xi) = \sum_{n=0}^{4} (n+1)! \, \alpha_n \beta_{n+1},$$

$$\phi_2(\xi) = \sum_{n=0}^{3} (n+2)! \, \alpha_n \beta_{n+2},$$

$$\phi_3(\xi) = \sum_{n=0}^{2} (n+3)! \, \alpha_n \beta_{n+3},$$

$$\phi_4(\xi) = \sum_{n=0}^{1} (n+4)! \, \alpha_n \beta_{n+4},$$

Clearly the coefficient $\beta_0 = 0$, since $\phi(\omega)$ and all its derivatives tend to zero as $\omega \to \infty$.

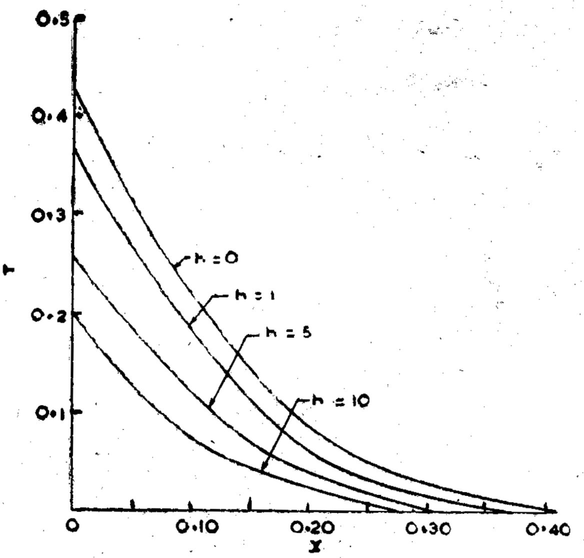

Fig 1—Temperature distribution at various time in the slab at $v=1/3$.

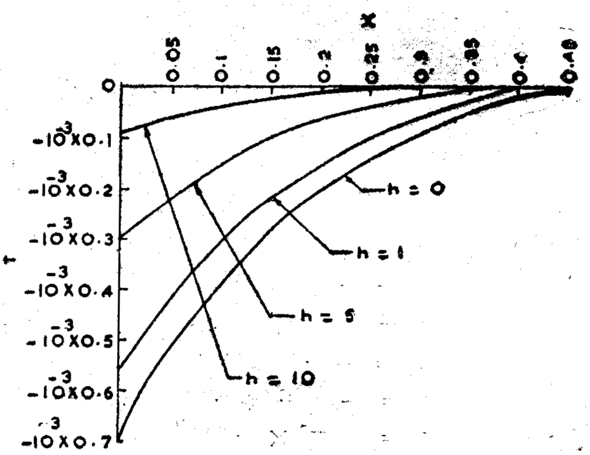

Fig 2—Temperature distribution at various time in the slab at $v=\frac{1}{2}$.

Now the expression (14) for T for values of ω near ξ takes the form

$$T = e^{-b\,|x|} \left[\sum_{n=0}^{\infty} \gamma_n\, q^n\right] \left[\sum_{n=1}^{5} \beta_n\,(\omega-\xi)^n\right] \tag{20}$$

Setting $\omega = \xi$, the general expressions for T becomes

$$T = e^{-b\,|x|} \sum_{n=1}^{5} n \mid \gamma_n\, \beta_n \tag{21}$$

It can be seen that the terms of this finite series decrease very rapidly.

DISCUSSION OF THE RESULTS OF EXAMPLES

In this section we discuss in detail the numerical examples considered in Case II of last section. For $v = 1/3$, $\omega = 1$, the first term in (17) is 30311043 $\times$ $e^{-b|u|}$ and the third term is $-0\cdot00000018 \times e^{-b|u|}$. Values of temperature T, with x at $v = 1/2$ and $1/3$ for various values of h ($h = 0,1,5,10$) are calculated and shown in Fig. 1 and 2 respectively.

From Figures following observations are made :

(*i*) The temperature zero when $u \to \infty$, $\omega \to \infty$.

(*ii*) The fall in temperature as u increase is very rapid whereas with the increase of time, the fall in temperature is slow.

(*iii*) At $v=\frac{1}{2}$, the temperature is negative but is quite small in magnitude.

(*iv*) It is interesting to note that the values of the temperature at $v=\frac{1}{3}$ is one-third the values at $v=1$.

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