TIME REVERSAL PROBLEM OF HEAT CONDUCTION IN SOLIDS OF SPHERICAL SYMMETRY

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(*Received* 10 *November* 1972 ; *revised* 9 *August* 1975)

In this paper the time reversal heat conduction problems for the spherical shells with heat generation and radiation with known boundary conditions have been solved by operational methods.

In 'Time Reversal Problems' we determine initial temperature distribution when the temperature distribution at any instant T $(T>0)$ is known to us. Sabherwal[1] has considered the time reversal problems in heat conduction for (i) semi-infinite medium, (ii) rectangular plate. Under the head of time reversal problems Mehta[2] has studied (i) heat flow in a cylindrical shell of infinite height with heat generation and radiation, (ii) heat flow in a truncated wedge of finite height, (iii) heat flow in a semi-infinite solid containing an exterior plane crack with a circular boundary and an infinitely long cylindrical cavity.

In this paper we have considered the flow of heat in a spherical shell and in a solid sphere with heat generation and radiation. The temperature distribution at a given time T is known and the initial temperature distribution is determined.

FLOW OF HEAT IN A SPHERICAL SHELL WITH HEAT GENERATION AND RADIATION

We consider here the flow of heat in a spherical shell $a \leqslant r \leqslant b$. Let the radiation take place at the surfaces $r=a$ and $r=b$, and temperature distribution $v_0(r)$ at time $t=T$. The heat source is within the shell. We assume that the temperature distribution depends upon the radial coordinate r and time t only. In this case the equation of conduction of heat is given[3] by

$$\frac{\partial v}{\partial t} = k\left[\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right] + Q(r, t) \tag{1}$$

where $v = v(r, t)$ is the temperature distribution at any instant t, k is the diffusivity constant and heat is supplied at the rate of $Q(r, t)$ per unit time per unit volume.

The appropriate physical conditions are given as

$$\left[\frac{\partial v}{\partial r} - k_1 v(r,t)\right]_{r=a} = f_a(t), \quad t > 0 \tag{2}$$

$$\left[\frac{\partial v}{\partial r} + k_1 v(r,t)\right]_{r=b} = f_b(t), \quad t > 0 \tag{3}$$

where k_1 is radiation constant whose value can be positive or zero.

$$v(r,t)\Big|_{t=0} = w(r), \quad \text{(unknown)} \qquad a < r < b \tag{4}$$

$$v(r,t)\Big|_{t=T} = v_0(r), \quad \text{(known)} \qquad a < r < b \tag{5}$$

Using the substitution

$$v(r,t) = r^{-\frac{1}{2}}\, u(r,t), \tag{6}$$

the equations (1) to (5) become

$$\frac{\partial u}{\partial t} = k \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{4r^2} u(r, t) \right] + Q_1(r, t) \tag{7}$$

where

$$Q_1(r, t) = r^{\frac{1}{2}} Q(r, t) \tag{8}$$

$$\left[r^{-\frac{1}{2}} \left\{ \frac{\partial u}{\partial r} - u \left(k_1 + \frac{1}{2r} \right) \right\} \right]_{r=a} = f_a(t), t > 0 \tag{9}$$

$$\left[r^{-\frac{1}{2}} \left\{ \frac{\partial u}{\partial r} + u \left(k_1 + \frac{1}{2r} \right) \right\} \right]_{r=b} = f_b(t), t > 0 \tag{10}$$

$$u(r, t) \Big|_{t=0} = r^{\frac{1}{2}} w(r) = w_1(r) \quad \text{(unknown)}, \quad a < r < b \tag{11}$$

$$u(r, t) \Big|_{t=T} = r^{\frac{1}{2}} v_0(r) = v_1(r) \quad \text{(known)}, \quad a < r < b \tag{12}$$

Cinelli[4] has defined finite Hankel transform as

$$\bar{f}(\xi_i) = \int_a^b r f(r) C_m(r, \xi_i) dr, a \leqslant r \leqslant b \tag{13}$$

where

$$C_m(r, \xi_i) = J_m(\xi_i r) \left[\xi_i Y'_m(\xi_i a) + h_1 Y_m(\xi_i a) \right] - $$
$$- Y_m(\xi_i r) \left[\xi_i J'_m(\xi_i a) + h_1 J_m(\xi_i a) \right] \tag{14}$$

$J_m(\xi_i r)$ and $Y_m(\xi_i r)$ are Bessel functions of the first kind and second kind respectively and of the order m and ξ_i is a root of the equation

$$\left[\xi_i Y'_m(\xi_i a) + h_1 Y_m(\xi_i a) \right] \left[\xi_i J'_m(\xi_i b) + h_2 J_m(\xi_i b) \right] =$$
$$= \left[\xi_i Y'_m(\xi_i b) + h_2 Y_m(\xi_i b) \right] \left[\xi_i J'_m(\xi_i a) + h_1 J_m(\xi_i a) \right] \tag{15}$$

Inversion theorem of (13) is

$$f(r) = \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 \left[\xi_i J'_m(\xi_i b) + h_2 J_m(\xi_i b) \right]^2 \bar{f}(\xi_i) \cdot \frac{C_m(r, \xi_i)}{F_m(\xi_i)} \tag{16}$$

where

$$F_m(\xi_i) = \left[h_2^2 + \xi_i^2 \left\{ 1 - \left(\frac{m}{\xi_i b} \right)^2 \right\} \right] \left[\xi_i J'_m(\xi_i a) + h_1 \cdot J_m(\xi_i a) \right]^2 -$$
$$- \left[h_1^2 + \xi_i^2 \left\{ 1 - \left(\frac{m}{\xi_i a} \right)^2 \right\} \right] \left[\xi_i J'_m(\xi_i b) + h_2 J_m(\xi_i b) \right]^2 \tag{17}$$

and the summation is taken over the positive roots of the equation (15)

The operational property of (13) is

$$\int_a^b r\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{m^2}{r^2}f\right]C_m(r,\xi_i)\,dr=\frac{2}{\pi}\Big[\alpha\{f'(b)+h_2f(b)\}-$$
$$-\{f'(a)+h_1f(a)\}\Big]-\xi_i^2\bar{f}(\xi_i) \qquad (18)$$

where

$$\alpha=\frac{[\xi_i J'_m(\xi_i a)+h_1 J_m(\xi_i a)]}{[\xi_i J'_m(\xi_i b)+h_2 J_m(\xi_i b)]} \qquad (19)$$

Applying (13) to (7) for the variable r and using (9), (10) and (18) we obtain

$$\frac{d\bar{u}}{dt}+k\,\xi_i^2\,\bar{u}(\xi_i,t)=\frac{2k}{\pi}\left[\alpha\, b^{\frac{1}{2}} f_b(t)-a^{\frac{1}{2}} f_a(t)\right]+\bar{Q}_1(\xi_i,t) \qquad (20)$$

where

$$\bar{Q}_1(\xi_i,t)=\int_a^b r^{3/2}\,Q(r,t)\,C_{\frac{1}{2}}(r,\xi_i)\,dr \qquad (21)$$

and

$$h_1=-\left(k_1+\frac{1}{2a}\right),\; h_2=\left(k_1-\frac{1}{2b}\right)\text{ and } m=\frac{1}{2} \qquad (22)$$

Solving (20) for $\bar{u}(\xi_i,t)$ and using (11), we get

$$\bar{u}(\xi_i,t)=\bar{w}_1(\xi_i)\,e^{-k\xi_i^2 t}+\frac{2k}{\pi}\int_0^t\Big[\alpha\, b^{\frac{1}{2}} f_b(y)-$$
$$-a^{\frac{1}{2}} f_a(y)+\frac{\pi}{2k}\bar{Q}_1(\xi_i,y)\Big]e^{-k\xi_i^2(t-y)}\,dy \qquad (23)$$

where

$$\bar{w}_1(\xi_i)=\int_a^b r^{3/2}\,w(r)\,C_{\frac{1}{2}}(r,\xi_i)\,dr \qquad (24)$$

From (12) and (23), the value of $\bar{w}_1(\xi_i)$ for $t=T$ is obtained as

$$\bar{w}_1(\xi_i)=\bar{v}_1(\xi_i)\,e^{k\xi_i^2 T}-\frac{2k}{\pi}\int_0^T\Big[\alpha\, b^{\frac{1}{2}} f_b(y)-$$
$$a^{\frac{1}{2}} f_a(y)+\frac{\pi}{2k}\bar{Q}_1(\xi_i,y)\Big]e^{k\xi_i^2 y}\,dy, \qquad (25)$$

where

$$\bar{v}_1(\xi_i)=\int_a^b r^{3/2}\,v_0(r)\,C_{\frac{1}{2}}(r,\xi_i)\,dr \qquad (26)$$

Using (4) and (16) we get the initial temperature distribution $w(r)$ as

$$w(r) = r^{\frac{1}{2}} \frac{\pi^2}{2} \sum_{\xi_i} \xi_i^2 \left[\frac{C_{\frac{1}{2}}(r, \xi_i)}{F_{\frac{1}{2}}(\xi_i)} \right] \left[\xi_i J'_{\frac{1}{2}}(\xi_i b) + \right.$$

$$\left. + h_2 J_{\frac{1}{2}}(\xi_i b) \right]^2 \left[\bar{v}_1(\xi_i) e^{k \xi_i^2 T} - \frac{2k}{\pi} \int_0^T \left\{ \alpha b^{\frac{1}{2}} f_b(y) - \right. \right.$$

$$\left. \left. - a^{\frac{1}{2}} f_a(y) + \frac{\pi}{2k} \bar{Q}_1(\xi_i, y) \right\} e^{k \xi_i^2 y} dy \right] \tag{27}$$

where $C_{\frac{1}{2}}(r, \xi_i)$ and $F_{\frac{1}{2}}(\xi_i)$ are given by (14) and (17) respectively with $m = \frac{1}{2}$.

The summation is taken over all the positive roots of the equation (15) with $m = \frac{1}{2}$.

FLOW OF HEAT IN A SOLID SPHERE WITH HEAT GENERATION AND RADIATION

The solid sphere is obtained by letting the inner radius of the spherical shell considered in the previous section approach zero. The temperature distribution $u(r,t)$ satisfies (7) and the physical conditions in this case are given by (10), (11) and (12).

In this case we use the transform given by Sneddon[5] as

$$\bar{f}(\xi_i) = \int_0^b r f(r) J_m(r \xi_i) \, dr, \; 0 \leqslant r \leqslant b, \; m > -\frac{1}{2} \tag{28}$$

where $f(r)$ is a continuous function and satisfies Dirichlet's conditions in $0 \leqslant r \leqslant b$ and ξ_i is a root of the transcendental equation

$$\xi_i J'_m(\xi_i b) + h_2 J_m(\xi_i b) = 0 \tag{29}$$

Inversion theorem of (28) is

$$f(r) = \frac{2}{b^2} \sum_{\xi_i} \frac{\xi_i^2 \bar{f}(\xi_i) J_m(r \xi_i)}{[h_2^2 + (\xi_i^2 - m^2/b^2)][J_m(b\xi_i)]^2}, \tag{30}$$

and the summation is taken over the positive roots of the equation (29)

The Operational property of (28) is

$$\int_0^b r \left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{m^2}{r^2} f \right] J_m(r \xi_i) \, dr = b J_m(\xi_i b) [f'(b) + h_2 f(b)] - \xi_i^2 \bar{f}(\xi_i) \tag{31}$$

Substituting

$$h_2 = k_1 - \frac{1}{2b}, \quad m = \frac{1}{2} \tag{32}$$

and applying (28) to (7) for the variable r and using (10) and (31), we obtain

$$\frac{d\bar{u}}{dt} + k \xi_i^2 \bar{u}(\xi_i, t) = k b^{3/2} J_{\frac{1}{2}}(\xi_i b) f_b(t) + \bar{Q}_1(\xi_i, t) \tag{33}$$

where

$$\bar{Q}_1(\xi_i, t) = \int_0^b r^{3/2} Q(r, t) J_{\frac{1}{2}}(r, \xi_i)\, dr \tag{34}$$

Solving (33) for $\bar{u}(\xi_i, t)$ and using (11), we get

$$\bar{u}(\xi_i, t) = \bar{w}_1(\xi_i) e^{-k\xi_i^2 t} + \int_0^t \left[k b^{3/2} J_{\frac{1}{2}}(\xi_i b) f_b(y) + \bar{Q}_1(\xi_i, y) \right] e^{-k\xi_i^2(t-y)} dy \tag{35}$$

where

$$\bar{w}_1(\xi_i) = \int_0^b r^{3/2} w(r) J_{\frac{1}{2}}(r, \xi_i)\, dr \tag{36}$$

From (35) and (12), the value of $w_1(\xi_i)$ for $t = T$ is obtained as

$$\bar{w}_1(\xi_i) = \bar{v}_1(\xi_i) e^{k\xi_i^2 T} - \int_0^T \left[k b^{3/2} J_{\frac{1}{2}}(\xi_i b) f_b(y) + \bar{Q}_1(\xi_i, y) \right] e^{k\xi_i^2 y} dy \tag{37}$$

where

$$\bar{v}_1(\xi_i) = \int_0^b r^{3/2} v_0(r) J_{\frac{1}{2}}(r, \xi_i)\, dy \tag{38}$$

Using (4) and (30), we get the initial temperature distribution $w(r)$ as

$$w(r) = r^{\frac{1}{2}} \frac{2}{b^2} \sum_{\xi_i} \frac{\xi_i^2 J_{\frac{1}{2}}(r\xi_i)}{\left[h_2^2 + \left(\xi_i^2 - \frac{1}{4b^2} \right) \right] \left[J_{\frac{1}{2}}(b\xi_i) \right]^2} \times$$

$$\times \left[\bar{v}_1(\xi_i) e^{k\xi_i^2 T} - \int_0^T \left\{ k b^{3/2} J_{\frac{1}{2}}(\xi_i b) f_b(y) + \bar{Q}_1(\xi_i, y) \right\} e^{k\xi_i^2 y} dy \right] \tag{39}$$

where the summation is taken over all the positive roots of the equation (29) having $m = \frac{1}{2}$.

PARTICULAR CASES

Here we mention some particular cases of special interest.

Case—I

Let the heat be generated at a constant rate A_0 per unit time per unit volume, the temperature distribution at time $t = T$ is taken unity and the radiation at the surface $r = b$ takes place into a medium of zero temperature.

Therefore, we have

$$v_0(r) = 1, \; f_b(t) = 0$$

and

$$Q(r, t) = (k / K) A_0$$

where K is the thermal conductivity of the material of the sphere.

Hence the initial temperature distribution in this case is obtained from the general result (39) as

$$w(r) = \frac{2r^{\frac{1}{2}}}{b^{\frac{1}{2}}} \sum_{\xi_i} \frac{\xi_i J_{\frac{1}{2}}(r\xi_i)}{\left[h_2^2 + \xi_i^2 - \frac{1}{4b^2} \right] \left[J_{\frac{1}{2}}(b\xi_i) \right]^2} \times$$

$$\times J_{3/2}(\xi_i b)\, e^{k\xi_i^2 T} \left[1 - \frac{A_0}{K\xi_1^2} \left(1 - e^{-k\xi_i^2 T} \right) \right]$$

Case—II

Assuming

$$v_0(r) = 1, f_b(t) = 0$$

and

$$Q(r,t) = \frac{A_0 k}{K} r^{\mu}, \quad \mu > 0$$

A_0 being constant, and using the result[6]

$$\int_0^a y^n J_m(y)\,dy = \frac{a^{n+1+m}}{2^m(n+1+m)\Gamma(m+1)} \cdot {}_1F_2\left[\frac{n+1+m}{2}; \frac{n+3+m}{2}, m+1; -\frac{a^2}{4}\right], \qquad \mathrm{Re}(m+n) > -1$$

in (39), we obtain $w(r)$ as

$$w(r) = 2(r/b)^{\frac{1}{2}} \sum_{\xi_i} \frac{\xi_i J_{\frac{1}{2}}(\xi_i r)\exp(k\xi_i^2 T)}{\left[h_2^2 + \xi_i^2 - \frac{1}{4b^2}\right]\left[J_{\frac{1}{2}}(\xi_i b)\right]^2} \times$$

$$\times \left[J_{3/2}(\xi_i b) - (2/\pi)^{1/2} \frac{A_0 b^{(3/2+\mu)}}{K \xi_i^{\frac{1}{2}}(3+\mu)}\, {}_1F_2\left\{\frac{3+\mu}{2}; \frac{5+\mu}{2}; \frac{3}{2}; -\frac{b^2\xi_i^2}{4}\right\}\left(1 - e^{-k\xi_i^2 T}\right)\right], \quad \mu > 0$$

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