STABILITY OF ELECTRICALLY CONDUCTING, ROTATING FLUID IN THE PRESENCE OF A HORIZONTAL MAGNETIC FIELD

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The Rayleigh instability of an incompressible, finitely conducting, inviscid fluid of variable density is investigated under the influence of an horizontal magnetic field. Due to finite conducting of the fluid, it is found that potentially stable or unstable configuration retains its character. It may further be noted that the stratification can be stabilized for a certain wave-number range in the absence of finite resistivity. The effect of finite resistivity is this to wipe out the aforesaid range of wave-number for which the system is stabilized otherwise.

The problem of stability of matter is of great importance in the peaceful uses of nuclear energy, specially in the controlled thermonuclear reactions. If the reaction is uncontrolled, it unleashes tremendous amount of energy instantaneously and this is the underlying principle behind the atom bomb. But if the reaction is to be controlled, one must take into account the possibility of the confinement. The basic question is whether the instability due to resistivity is tolerable or untolerable. In the present paper we have tried to find out to what extent the instability of the configuration is affected by changing the resistivity of the system.

Lord Rayleigh[1] was one of the first to initiate the study of the hydrodynamic stability of a stratified inviscid fluid of variable density, he found that the equilibrium of a horizontal layer of a heavy incompressible fluid of a variable density ρ_0 is stable or unstable according as $\frac{d\rho_0}{dz}$ is everywhere negative or anywhere positive. Chandrasekhar[2] introduced the viscosity into such problem and Hide[3] further studied the case of a viscous conducting fluid with a transverse magnetic field. It was found that magnetic field considerably stabilizes the configuration and it is possible to have oscillatory motion in the presence of magnetic field even if the configuration is throughly unstable.

Hide[4] also considered the effect of rotation on the character of the equilibrium of a stratified hetrogenous inviscid fluid, and found that rotation stabilizes the potentially unstable arrangement of certain wave number.

The problem of the hydromagnetic stability of conducting fluid of variable density plays important role in the study of astrophysics (theories and sunspot magnetic fields, heating of solar corona, stability of the steller atmosphere in magnetic field). Since in the astrophysical problems, the coriolis forces play an important role, therefore, it is necessary to study the combined effect of rotation and magnetic field. Talwar[5] discussed for the first time the effect of a magnetic field (Horizontal) on the equilibrium of an inviscid, incompressible, infinitely conducting rotating fluid of variable density. Ariel[6] further studied the character of equilibrium of a heavy, viscous, incompressible, infinitely conducting, rotating fluid in the presence of a magnetic field (vertical). In both cases, it is found that both magnetic field and the coriolis forces tend to stabilize the configuration.

In a more realistic physical situation specially relevant to the problem of confinement of matter one must take into account the finite resistivity of the medium. The present paper is an attempt in that direction. We, thus investigate the effect of a horizontal magnetic field on the equilibrium an inviscid, incompressible, finitely conducting rotating fluid of variable density.

BASIC EQUATIONS

The linearised perturbation equations for the problem under consideration are

$$\rho_0 \frac{\partial \underline{u}}{\partial t} + \rho_0 (\underline{u}. \nabla) \underline{u} = - \nabla \delta p + \frac{\kappa}{4\pi} (\nabla \times \underline{h}) \times \underline{H} + 2 \rho_0 (\underline{u} \times \Omega + g\delta\rho , \tag{1}$$

$$\frac{\partial}{\partial t}\,\delta\rho + \underline{u}\cdot\nabla\,\rho_0 = 0\,,$$

$$\frac{\partial h}{\partial t} = H_0\,\nabla\,\underline{u} + \eta\,\nabla^2\,\underline{h}\,, \tag{3}$$

$$\nabla\cdot\underline{u} = 0 \text{ and } \nabla\cdot\underline{h} = 0\,, \tag{4}$$

where ρ_0, p_0 and H_0 denote respectively the density, pressure and magnetic field at a point. The latter being taken uniform and along the horizontal direction. The fluid is assumed to partake in uniform rotation Ω about z-axis. g is acceleration due to gravity (components 0, 0,$=g$), κ and η denote the coefficient of magnetic permeability and electrical resistivity at the medium. Finally $\delta\,\rho$, $\delta\,p$ and h denote the perturbations in density, pressure and magnetic field respectively consequent to a small disturbance which produces a velocity field ($\underline{u}= u, v, w$).

Analysing the disturbance into normal modes, we seek solutions whose dependence on x, y and t is given by exp $(ik_x\,x + ik_y\,y + nt)$, where k_x and k_y are the horizontal components of the wave vector $\underline{k}$ and n is the rate at which the system departs from equilibrium, we find that the z-components of curl and curl curl of equation (1), and equation (3) and its curl take the form.

$$n\left[\,k^2\,\rho_0\,w - D\,\rho_0\,D\,w)\,\right] - \frac{gk^2}{n}\,(D\,\rho_0)\,w + \frac{\kappa}{4\pi}\,ik_x\,(D^2 - k^2)\,h_z - 2\,\Omega\,D\,(\rho_0\,\zeta) = 0\,, \tag{5}$$

$$n\,\rho_0\,\zeta - \frac{\kappa\,H_0}{4\pi}\,i\,k_x\,\xi = 2\,\rho_0\,\Omega\,D\,w\,, \tag{6}$$

$$\left[\,n - \eta\,(D^2 - k^2)\,\right]\,h_z = ik_x\,H_0\,w\,, \tag{7}$$

$$\left[\,n - \eta\,(D^2 - k^2)\,\right]\,\xi = ik_x\,H_0\,\zeta\,, \tag{8}$$

where

$$\zeta = ik_x\,v - ik_y\,u\,, \tag{9}$$

and

$$\xi = ik_x\,h_y - ik_y\,h_x\,, \tag{10}$$

are the z-components of curl $\underline{u}$ and curl $\underline{h}$ respectively.

BOUNDARY CONDITIONS

We assume that the fluid in confined between two rigid planes $z = 0$ and $z = d$, since the normal velocity at a boundary surface vanishes hence, we have

$$w = 0 \tag{11}$$

at a rigid boundary.

For electromagnetic boundary conditions, we see that if the fluid in bounded by an ideal conductor, no disturbance within the fluid can charge E and H outside the fluid. Since surface charges and surface currents can allow discontinuities in E_z, h_x and h_y, we must require that

$$h_z = E_x = E_y = 0 \tag{12}$$

which leads to

$$D\,\xi = 0 \text{ and } h_z = 0 \tag{13}$$

at a surface boundary by an ideal conductor.

We do not require a boundary condition on ζ for the value of ζ can be obtained by substituting the value of ξ in equation (8).

A VARIATIONAL PRINCIPLE

Multiplying the equation (5) for the characteristic value n_i by w_j and integrating across the vertical extent of the fluid, we obtain the following equation, after a series of integrations by parts.

$$n_i \int_0^d \rho_0 (k^2 w_i w_j + D w_i D w_j) dz - \frac{g k^2}{n_i} \int_0^d D \rho_0 w_i w_j dz - \frac{\kappa k^2}{4\pi} (\eta_j + \eta k^2) \int_0^d h_i h_j dz -$$

$$- \frac{\kappa}{4\pi} (\eta_j + 2 \eta k^2) \int_0^d D h_i D h_j dz - \frac{\kappa}{4\pi} \frac{\eta k^2}{k^2} \int_0^d D^2 h_i D^2 h_j dz + \eta_j \int_0^d \rho_0 \zeta_i \zeta_j dz -$$

$$- \frac{\kappa}{4\pi} (n_i + \eta k^2) \int_0^d \xi_i \xi_j dz - \frac{\kappa}{4\pi} \frac{\eta k^2}{k^2} \int_0^d D \xi_i D \xi_j dz = 0, \qquad (14)$$

the integrated parts vanish on account of boudary conditions.

Setting $i = j$ in equation (14), we get

$$n (I_1 - I_3 - I_4 + I_6 - I_7) - (g k^2/n) I_2 - \eta k^2 (I_3 + 2 I_4 + I_5 + I_7 + I_8) = 0 \qquad (15)$$

where

$$I_1 = \int_0^d \rho_0 \left[k^2 w^2 + (Dw)^2 \right] dz, \quad (16) \qquad I_2 = \int_0 w^2 D \rho_0 dz, \quad (17)$$

$$I_3 = \frac{\kappa k^2}{4\pi} \int_0^d h^2 dz, \quad (18) \qquad I_4 = \frac{\kappa}{4\pi} \int_0^d (D h)^2 dz, \quad (19)$$

$$I_5 = \frac{\kappa}{4\pi k^2} \int_0^d (D^2 h)^2 dz, \quad (20) \qquad I_6 = \int_0^d \rho_0 \zeta^2 dz, \quad (21)$$

$$I_7 = \frac{\kappa}{4\pi} \int_0^d \xi^2 dz, \quad (22) \qquad I_8 = \frac{\kappa}{4\pi k^2} \int_0^d (D \xi)^2 dz. \quad (23)$$

Now by considering a small change δn in n consequent upon first order arbitrary variation δw, δh, $\delta \zeta$ and $\delta \xi$ in w, h, ζ and ξ respectively which satisfy the boundary conditions of the problem, we can show, by proceeding along standard lines, that a necessary and sufficient condition for δn to vanish is that w, h, etc. be solutions of the characteristic value problem. This provides the basis for obtaining an approximate solution of the problem by the variational procedure.

THE CASE OF EXPONENTIALLY VARYING DENSITY

A case for which a simple analytical solution can be found is one in which the undisturbed density distribution is given by

$$\rho_0 (z) = \rho_1 \exp \beta z, \qquad (24)$$

where ρ_1 and β are constants.

A further assumption, namely

$$| \beta d | << 1 \qquad (25)$$

is made, implying that the density variation within fluid is much less than the average density and therefore, has a negligible effect on the inertia of the fluid.

We shall now consider the case of the fluid confined between two rigid boundaries $z = 0$ and $z = d$ which are both ideally conducting.

Since ;

$$\left. \begin{aligned} w(0) &= w(d) = 0 \\ h(0) &= h(d) = 0 \\ D\xi(0) &= D\xi(d) = 0 \end{aligned} \right\} \qquad (26)$$

We assume the following trial function for $w(z)$, $h(z)$, and $\xi(z)$ respectively

$$w(z) = W \sin lz \qquad (27)$$

$$h(z) = K \sin lz \qquad (28)$$

$$\xi(z) = X \cos lz \tag{29}$$

where $l = \pi s/d$, s being any integer.

The value of $\zeta(z)$ now can be obtained by inserting the value of $\xi(z)$ in euqation (8), we have

$$\zeta(z) = Z \cos lz, \tag{30}$$

where

$$\left[n + \eta \; (l^2 + k^2) \right] X = ik_x H_0 Z \tag{31}$$

Substituting $w(z)$, $h(z)$, $\zeta(z)$ and $\xi(z)$ in (6) and (7) respectively, we obtain the following equations.

$$n Z - \frac{\kappa H_0}{4\pi \rho_1} ik_x X = 2\,\Omega\, l\, W, \tag{32}$$

and

$$\left[n + \eta\,(l^2 + k^2) \right] K = n_2 K = ik_x H_0 W. \tag{33}$$

Solving the equations (31), (32) and (33), we get

$$K = ik_x H_0 W/n_2,\; X = 2\,\Omega\, H_0\, ik_x\, W/(nn_2 + V_0^2 k_x^2)\;,\; Z = 2\,\Omega\, l\, n_2\, W/nn_2 + V_0^2 k_x^2) \tag{34}$$

where V_0 denotes the so called Alfven velocity given by

$$V_0^2 = H_0^2/4\pi \rho_1. \tag{35}$$

Substituting the values of trial functions (27) to (29) in the variational formula (15), we obtain the following dispersion relations between n and k, after eliminating K^2/W^2, X^2/W^2, Z^2/W^2 and on making use of equations listed in (34)

$$\begin{aligned} n^5 + 2n^4 \eta\,(l^2 + \;\;^2) + n^3 \left[\eta^2 (l^2 + k^2)^2 + 2V_0^2 k^2 \cos^2\theta + \frac{4\,\Omega^2 l^2 - g\beta k^2}{l^2 + k^2} \right] + \\ + 2n^2 \eta \left[4\,\Omega^2 l^2 - g\beta k^2 + V_0^2 k^2 \cos^2\theta\,(l^2 + k^2) \right] + n \left[\eta^2 (l^2 + \right. \\ \left. + k^2)\,(4\,\Omega^2 l^2 - g\beta k^2) + V_0^4 k^4 \cos^4\theta - V_0^2 k^2 \cos^2\theta \;\frac{g\beta k^2}{l^2 + k^2} \right] - \\ - 2g\beta\eta\, k^4 V_0^2 \cos^2\theta = 0\,, \end{aligned} \tag{36}$$

θ being the inclination of the direction of wave-vector to that of the magnetic field.

It is reduced to the corresponding dispersion relation obtained by Talwar[4] when $\eta = 0$.

It is convenient to discuss equation (36) in non-dimensional form, we choose dimensionless growth rate y and a dimensionless, wave-number x by measuring n and k in suitable units, by defining

$$x = kd/\pi\, s, \text{ and } y = nd/\pi\, sV_0\,. \tag{37}$$

From equations (36) and (37), we have

$$\begin{aligned} y^5 + 4y^4 R\,(1 + x^2) + y^3 \left[4R^2 (1 + x^2)^2 + 2x^2 \cos^2\theta + \frac{A - Bx^2}{1 + x^2} \right] + 4y^2 R \left[(A - Bx^2) + \right. \\ \left. + x^2 (1 + x^2) \cos^2\theta \right] + y \left[4R^2 (1 + x^2)\,(A - Bx^2) + x^4 \cos^4\theta - \frac{Bx^4 \cos^2\theta}{1 + x^2} \right] - \\ - 2BRx^4 \cos^2\theta = 0\,, \end{aligned} \tag{38}$$

where

$$A = 4\,\Omega^2 d^2/\pi^2 s^2 V_0^2\,, \tag{39}$$

$$B = g\beta d^2/\pi^2 s^3 V_0^2\,, \tag{40}$$

and

$$R = \pi\,\eta\, s/2\,V_0\, d \tag{41}$$

From above equation, we see that, there are three parameters required to specify y for any given x. These parameters A, B and R represent measures of coriolis forces, buoyancy forces, and electrical resistivity in terms of magnetic field, respectively.

Equation (38) is a quantic in y, hence it will have five roots. It is too difficult to solve it explicity for arbitrary value of A, B, R and x. However, we can draw a few general conclusions. It can be seen that if the absolute term in equation (38) is positive that is $B < 0$, by Hurwitz criterion equation (38) does not admit any positive root and the equilibrium is always s able. Thus the potentially stable arrangement remains intact.

If $B > 0$, the absolute term in equation (38) is negative therefore, the equation (38) being of an odd degree must have necessarily at least one real positive root. In fact, it is the only positive root and corresponding to this root equilibrium is always unstable. It may be further noted that the stratification can be stabilized for a certain wave-number range ($x > \sqrt{B-1}$) in the absence of finite resistivity. The effect of finite resistivity is thus to wipe out the aforesaid range of wave-number for which the system is stabilized otherwise.

We shall now discuss the nature of this positive root y in detail. The asymptotic behaviour of this root for $x \to 0$ and $x \to \infty$ are

$$y \to B x^4 \cos^2 \theta / 2 A R \,, \; (x \to 0), \tag{42}$$

and

$$y \to [(\cos^4 \theta + 16 B R^2)^{\frac{1}{2}} - \cos^2 \theta]/4R. \; (x \to \infty) \tag{43}$$

We shall now consider the behaviour of y on varying the value of R, the measure of electrical resistivity. A peculiar tendency is exhibited by y as we vary R. Two cases arise (*a*) $R > R^*$ (*b*) $R < R^*$

where

$$R^* = \frac{1}{2} \sqrt{\frac{B}{A+B}} \cos \theta \tag{44}$$

Whereas in the former case no mode of maximum instability occurs—y monotonically increases from zero and approaches $\dfrac{(\cos^4 \theta + 16 B R^2)^{\frac{1}{2}} - \cos^2 \theta}{4R}$ asymptotically; in the latter case there is always a mode of maximum instability. Thus the electrical resistivity suppresses the mode of maximum instability for a reasonably large value of R.

It can be further noted from Fig. 1 that an increase in the value of R leads to a decrease in the value of y for small values of x, but to an increase in the values of y for large values of x. Thus, we can conclude that electrical resistivity has a stabilizing influence for small wave-number (or large wave-lengths) of disturbance but it has a destabilizing effect for large wave-numbers (or small wave-lengths).

The calculation of the positive root of the equation (38) has also been carried out for $B = 5, R = 1$ and for several values of A, and the results are presented graphically in Fig. 2. Curves of y against x for $B = 5$, $R = 1$ and $A = 0, 10, 100$ are plotted. These curves clearly show that no mode of maximum instability exists. (Since in this case $R > R^*$ for all values of A and B). Further the curves of y against x for $R = 0 \cdot 2$, $B = 5$ and various values of A are plotted in Fig. 3. It can be seen (*i*) that for a given x, y decreases with increase of A (*ii*) that the maximum growth rate, also decrease with increase of A (*iii*) that, the wave-number for the mode of m ximum instability increases with increasing A, the parameter A being a measure of the relative dynamical importance of the coriolis forces with the magnetic forces. We can say for

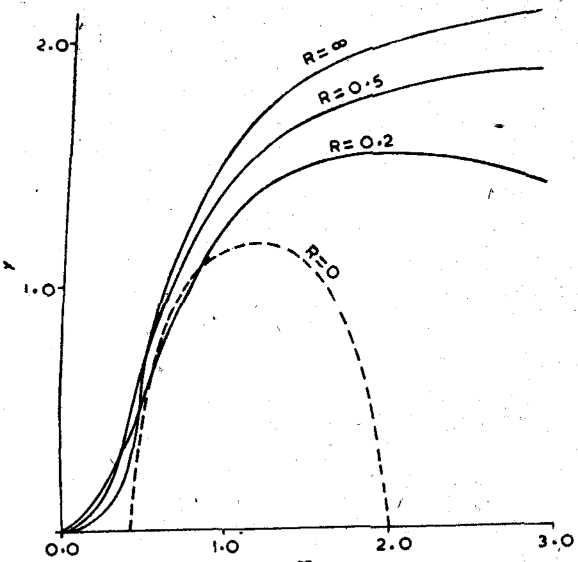

Fig. 1—The growth rate y plotted as a function of wave-number x for $B = 5$ and $A = 1$. The values of $R = \infty$, $0 \cdot 5$, $0 \cdot 2$ and 0 ($\theta = 0$).

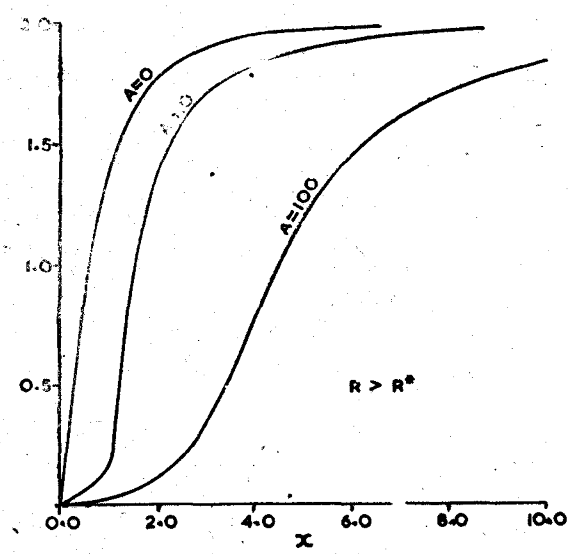

Fig. 2—The growth rate y plotted as a function of wave-number x for $B = 5$, and $R = 1$. The values of A are 0, 10 and 100 ($\theta = 0$).

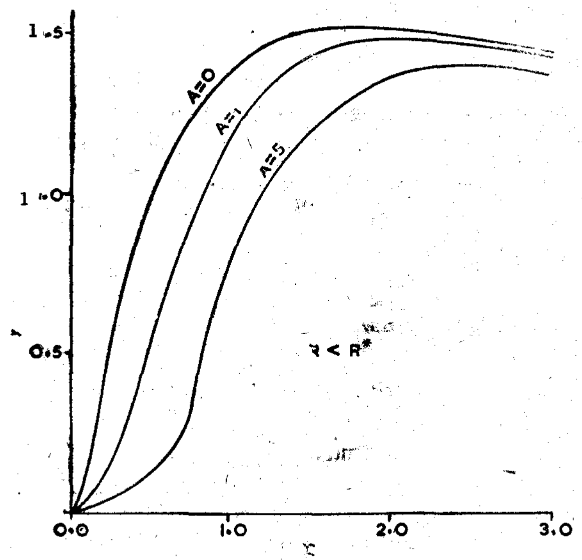

Fig. 3—The growth rate y plotted as a function of wave-number x for $B = 5$ and $R = 0{\cdot}2$. The values of A are 0, 1 and 5 ($\theta = 0$).

this case that, for a given B, the effect of increase in rotation is to increase the time taken for the system of depart from equilibrium and to decrease the wave-length of the mode of maximum instability.

WAVES IN THE ABSENCE OF BUOYANCY FORCES

Putting $B = 0$ in equation (38), we get for $\theta = 0$

$$\left[y^2 + 2\,y\,R\,(1 + x^2) + x^2 \right]^2 = \frac{A}{1 + x^2} \cdot \left[y + 2R\,(1 + x^2) \right]^2. \qquad (45)$$

If $A = 0$, the equation (45) becomes

$$y = -R\,(1 + x^2) \pm \left[R^2\,(1 + x^2)^2 - x^2 \right]^{\frac{1}{2}}. \qquad (46)$$

From equation (46) it can be seen that for $R > \frac{1}{2}$ (in general $R > \frac{1}{2} \cos\theta$) the motion is aperiodical for entire range of x, however, if $R < \frac{1}{2}$ the damped oscillations take place within the wave-range $x_1 < x < x_2$ (47)

where

$$x^2_{\;1,2} = \frac{1}{2R^2} \left[2R^2 - 1 \pm (1 - 4R^2)^{\frac{1}{2}} \right] \qquad (48)$$

Simplifying equation (45) further, we get

$$y^2 + 2y \left[R\,(1 + x^2) \pm \tfrac{1}{2}\, i \left(\frac{A}{1 + x^2} \right)^{\frac{1}{2}} \right] + x^2 \pm 2\,i\,R\,(1 + x^2)^{\frac{1}{2}} A^{\frac{1}{2}} = 0. \qquad (49)$$

The solution of equation (49) can be easily written

$$y = -\left[R\,(1 + x^2) \pm \tfrac{1}{2}\, i \left(\frac{A}{1 + x^2} \right)^{\frac{1}{2}} \right] \pm \left[R^2\,(1 + x^2)^2 - \frac{A}{4\,(1 + x^2)} - x^2 \mp A^{\frac{1}{2}}\,(1 + x^2)^{\frac{1}{2}} R \right]^{\frac{1}{2}}. \qquad (50)$$

Since in equation (50) the coefficient of y is complex with non-zero real part, it should have necessarily complex roots. Hence we find that rotation in the absence of buoyancy forces gives rise to damped oscillatory motion throughout the range of wave-number x, in contrast to the case when the coriolis forces are absent. Because in the latter case there is always a range of values of x for which the motion is aperiodically damped.

I (y), the angular frequency of oscillations, is given by

$$I\,(y) = \pm \tfrac{1}{2} \left(\frac{A}{1 + x^2} \right)^{\frac{1}{2}} \pm \frac{1}{2} \left\{ \left[\left(R^2\,(1 + x^2)^2 - \frac{A}{4\,(1 + x^2)} - x^2 \right)^2 - A\,(1 + x^2)\,R \right]^{\frac{1}{2}} - \left(R^2\,(1 + x^2) + \frac{A}{4\,(1 + x^2)} + x^2 \right) \right\}^{\frac{1}{2}}. \qquad (51)$$

The positive and negative signs taken in possible combinations give the angular frequency of four normal modes of oscillations. The expressions for the phase and group velocities can be obtained from equation (51), with the help of the following relations.

$$U_{p,x} = \pm\, I\,(y)\,,\; U_{q,x} = d\,I\,(y)/dx\,. \qquad (52)$$

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