LARGE DEFLECTION OF A HEATED ELLIPTICAL PLATE UNDER STATIONARY TEMPERATURE

PARITOSH BISWAS

P. D. Women's College, Jalpaiguri

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The large deflection of a heated elliptic plate with clamped edges using Berger's method has been investigated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind of order $2m$.

In the postwar years we have seen a rapid development of thermo-elasticity stimulated by various engineering sciences. A considerable progress in the field of aircraft and machine structures, mainly with gas and steam turbines, and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stresses play an important and frequently even a primary role.

As thermo-elasticity problems determination of thermal deflections of plates, especially for thin plates, is of vital importance in air-craft structures and in the design of machine parts, in as much as for thin plates there may be excessive deflections and consequently heavy thermal stresses may be developed and as a result there may not be proper functioning.

The classical large deflection plate problems usually lead to non-linear differential equations which cannot be exactly solved. Berger[1] has shown that if, in deriving the differential equations from the expressions for strain energy, the strain energy due to second invariant in the middle plane of the plate is neglected, a simple fourth order differential equation, coupled with a non-linear second order equation, is obtained. Although no complete explanation of the method is set forth, the stresses and deflections obtained by Berger himself for rectangular and circular plates agree well with those found from more precise analysis. This approximate method has been extended to orthotropic plates by Iwinski and Nowinski[2] and further boundary value problems associated with rectangular and circular plates have been solved by Nowinski[3]. The above technique of Berger has been used quite elegantly by Thein Wah and Robert Schmidt[4] and Nash & Modeer[5] to obtain satisfactory results.

Basuli[6] has extended this approximate method of Berger to problems under uniform load and heating under stationary temperature distribution.

In this paper the author has applied the method devised by Berger and Basuli to investigate the large deflection of an elliptic plate heated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind of order $2m$.

NOTATIONS

$$D = \frac{Eh^3}{12(1-\nu^2)} = \text{flexural rigidity of the plate},$$

h = thickness of the plate,

∇^2 = Laplacian operator,

E, ν, α_t = Young's modulus, Poisson's ratio and coefficient of thermal expansion,

Q = uniform load,

ω = lateral displacement,

u, v = displacements along the x-and y-axes,

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2,$$

$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial \omega}{\partial y}\right)^2,$$

$$e = e_{xx} + e_{yy} = \text{first invariant},$$

$$e_2 = e_{xx}\, e_{yy} - \frac{1}{4}\, e_{xy}^2 = \text{second invariant},$$

$K, K_1, K_2, C_{2m}, \overline{C}_{2m}, D_{2m}, a_{2m}$ = constants,

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial x} \cdot \frac{\partial \omega}{\partial y}$$

GOVERNING EQUATIONS

Combining the strain energy due to bending and stretching of the middle surface of the plate loaded normally without temperature and the strain energy due to heating, the total potential energy, is given by[7,10]

$$V = \iint_s \left[\frac{D}{2} \left\{ (\nabla^2 \omega)^2 + \frac{12}{h^2} e^2 - 2(1-\nu)\left[\frac{12}{h^2} e_2 + \frac{\partial^2 \omega}{\partial x^2} \cdot \frac{\partial^2 \omega}{\partial y^2} - \left(\frac{\partial^2 \omega}{\partial x \partial y} \right)^2 \right] \right\} - \right.$$

$$\left. - Q\omega \right] dxdy - \iint_s \int_{-h/2}^{h/2} \frac{E\alpha_t}{(1-\nu)} T'(x, y, z)\,[e - z\nabla^2 \omega]\, dxdydz \tag{1}$$

where the symbol $\iint_s$ indicates integration over the surface s of the plate.

Let the temperature distribution $T'(x, y, z)$ be assumed in the form[6]

where $$T'(x, y, z) = T_0(x, y) + g(z)\, T(x, y) \tag{2}$$

$$\int_{-h/2}^{h/2} z\, g(z)\, dz = f(h) \text{ and } \int_{-h/2}^{h/2} g(z)\, dz = 0 \tag{3}$$

Combining (1), (2) and (3) the equation (1) simplifies into the form

$$V = \iint_s \left[\frac{D}{2} \left\{ (\nabla^2 \omega)^2 + \frac{12}{h^2} e^2 - 2(1-\nu)\left[\frac{12}{h^2} e_2 + \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 \omega}{\partial y^2} - \left(\frac{\partial^2 \omega}{\partial x \partial y} \right)^2 \right] \right\} - \right.$$

$$\left. - Q\omega - \frac{E\alpha_t}{1-\nu} \left\{ T_0\, eh - T f(h)\, \nabla^2 \omega \right\} \right] dxdy \tag{4}$$

Neglecting e_2 and using Euler's variational equations, the following differential equations are obtained[6]

$$\nabla^2 (\nabla^2 - \beta_1^2)\, \omega = \frac{1}{D} \left\{ Q - \frac{E\alpha_t f(h)}{1-\nu} \nabla^2 T \right\} \tag{5}$$

$$e - (1+\nu)\,\alpha_t T_0 = \frac{\beta_1^2 h^2}{12} \tag{6}$$

where β_1^2 is a normalised constant of integration.

ANALYSIS

Let us take an elliptic plate of thickness h. We take the centre of the plate in the middle surface as the origin and z-axis downwards.

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution[8]

$$\nabla^2 T_0 - \epsilon T_0 = - \frac{\epsilon_0}{2} \left(\theta_1 + \theta_2\right) \tag{7}$$

$$\nabla^2 T - \frac{12}{h^2} \left(1 + \epsilon\right) T = - \frac{12 \epsilon}{h^3} \left(\theta_1 - \theta_2\right) \tag{8}$$

where θ_1 and θ_2 denote temperatures at the upper and lower media of the plate respectively. The equation (8) can be put in the particular form

$$\nabla^2 T - C^2 T = 0 \tag{9}$$

where

$$C^2 = \frac{12}{h^2}\left(1 + \epsilon\right)$$

Transferring to elliptic coordinates (ξ, η) defined by $x + iy = d \cosh (\xi + i\eta)$, where $2d$ is the interfocal distance of the ellipse, the equation (9) reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{C^2 d^2}{2} \left(\cosh 2\xi - \cos 2\eta\right) T = 0 \tag{10}$$

The solution of (10) can be found in the form

$$T(\xi, \eta) = \sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi, -q)\, ce_{2m}(\eta, -q) \tag{11}$$

where $Ce_{2m}(\xi, -q)$ and $ce_{2m}(\eta, -q)$ are the modified Mathieu function and ordinary Mathieu function of the first kind of order $2m$ and

$$q = C^2 d^2/4$$

Imposing the boundary condition

T = Constant = K on $\xi = \xi_0$ one gets from (11)

$$K = \sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi_0, -q)\, ce_{2m}(\eta, -q) \tag{12}$$

Multiplying both sides of (12) by $ce_{2m}(\eta, -q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation[9], one gets

$$C_{2m} = 2 K A_0^{2m} / Ce_{2m}(\xi_0, -q) \tag{13}$$

where A_0^{2m} is the Fourier co-efficient in the expansion of

$$Ce_{2m}(\eta, -q).$$

Also to solve equation (7), we suppose

$$\theta_1 + \theta_2 = \text{constant} \tag{14}$$

and write the equation in the form

$$\nabla^2 T_0 - K_1^2 T_0 = K_2 \tag{15}$$

where

$$\epsilon = K_1^2 \text{ and } - \frac{\epsilon_0}{2} (\theta_1 + \theta_2) = K_2 \tag{16}$$

Transferring the equation (15) into elliptic coordinates, the general solution is obtained as

$$T_0(\xi,\eta) = \sum_{m=0}^{\infty} a_{2m}\, Ce_{2m}(\xi, -q_1)\, ce_{2m}(\eta, -q_1) - \frac{K_2}{K_1^2} \tag{17}$$

where

$$q_1 = K_1^2\, d^2/4 \tag{18}$$

Let us assume the following boundary condition for T_0

$$T_0 = 0 \text{ at } \xi = \xi_0 \tag{19}$$

Substituting (19) into (17), one gets

$$\frac{K_2}{K_1^2} = \sum_{m=0}^{\infty} a_{2m}\, Ce_{2m}(\xi_0, -q_1)\, ce_{2m}(\eta, -q_1) \tag{20}$$

Multiplying both sides of (20) by $ce_{2m}(\eta, -q_1)$ and integrating with respect to η from 0 to 2π and using orthogonality relation and normalisation, one gets

$$a_{2m} = \frac{2\,(K_2/K_1^2)\,\bar{A}_0^{2m}}{Ce_0(\xi_0, -q_1)} \tag{21}$$

where $\bar{A}_0^{2m}$ is the Fourier co-efficient in the expansion of

$$ce_{2m}(\eta, -q_1)$$

Taking the load function Q equal to zero, equation (5) takes the form

$$\nabla^2(\nabla^2 - \beta_1^2)\,\omega = -\lambda\,\nabla^2 \sum_{m=2}^{\infty} C_{2m}\, Ce_{2m}(\xi, -q)\, ce_{2m}(\eta, -q) \tag{22}$$

where

$$\lambda = \frac{E\alpha_t f(h)}{D(1-\nu)} \tag{23}$$

For complementary function of equation (22), we assume

$$\omega = \omega_1 + \omega_2 \text{ such that } \nabla^2 \omega_1 = 0 \text{ and } \nabla^2 \omega_2 - \beta_1^2 \omega_2 = 0$$

Changing to elliptic coordinates the above equations reduce to

$$\frac{\partial^2 \omega_1}{\partial \xi^2} + \frac{\partial^2 \omega_1}{\partial \eta^2} = 0 \tag{24}$$

$$\frac{\partial^2 \omega_2}{\partial \xi^2} + \frac{\partial^2 \omega_2}{\partial \eta^2} - \frac{\beta_1^2 d^2}{2}\left(\cosh 2\xi - \cos 2\eta\right)\omega_2 = 0 \tag{25}$$

Periodic solutions of (24) and (25) which are symmetric about the centre can be represented by

$$\omega_1 = \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi \cos 2m\eta \tag{26}$$

$$\omega_2 = \sum_{m=0}^{\infty} D_{2m}\, Ce_{2m}(\xi, -q')\, ce_{2m}(\eta, -q') \tag{27}$$

where

$$q' = \beta_1^2\, d^2/4$$

Now a particular integral of

$$\left(\nabla^2 - \beta_1^2\right)\omega = -\lambda \sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi, -q)\, ce_{2m}(\eta, -q) \tag{28}$$

is necessarily a particular solution of equation (22). Clearly the particular integral of equation (28) is given by

$$\omega = -\frac{\lambda}{C^2-\beta_1^2}\sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi,-q)\, ce_{2m}(\eta,-q) \tag{29}$$

The general solution of equation (22) is therefore given by

$$\omega = \sum_{m=0}^{\infty} \bar{C}_{2m}\cosh 2m\xi\cos 2m\eta + \sum_{m=0}^{\infty} D_{2m}\, Ce_{2m}(\xi,-q')\, ce_{2m}(\eta,-q') - \frac{\lambda}{C^2-\beta_1^2}\sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi,-q)\, ce_{2m}(\eta,-q) \tag{30}$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, we have

$$\omega = \frac{\partial\omega}{\partial\xi} = 0 \quad \text{when } \xi = \xi_0$$

Using the above boundary conditions in (30) and multiplying the resulting relations by $ce_{2m}(\eta,-q)$ and integrating with respect to η from 0 to 2π one gets, after using normalisation, the constants $\bar{C}_{2m}$ and D_{2m} in the forms

$$\bar{C}_{2m} = \frac{\frac{\lambda}{C^2-\beta_1^2}C_{2m}\left\{Ce_{2m}(\xi_0,-q)\,ce'_{2m}(\xi_0,-q') - Ce'_{2m}(\xi_0,-q)\,Ce_{2m}(\xi_0,-q')\right\}}{A_{2m}^{2m}\left\{Ce'_{2m}(\xi_0,-q)\cosh 2m\xi_0 - 2m\,Ce_{2m}(\xi_0,-q')\sinh 2m\xi_0\right\}} \tag{31}$$

$$D_{2m} = \frac{\frac{\lambda}{C^2-\beta_1^2}C_{2m}\left\{2m\sinh 2m\xi_0\,Ce_{2m}(\xi_0,-q) - Ce'_{2m}(\xi_0,-q)\cosh 2m\xi_0\right\}}{\Phi\left\{2m\sinh 2m\xi_0\,Ce_{2m}(\xi_0,-q') - \cosh 2m\xi_0\,Ce'_{2m}(\xi_0,-q')\right\}} \tag{32}$$

where

$$\Phi = 2A_0^{2m}A_0'^{2m} - A_{2m}^{2m}A'^{2m}_{2m} \tag{33}$$

A_{2m}^{2m} and A'^{2m}_{2m} are the Fourier co-efficients in the expansions of $ce_{2m}(\eta,-q)$ and $ce_{2m}(\eta,-q')$ respectively.

To determine the constant, β_1^2, equation (6) is transformed to elliptic coordinates in the form

$$h_1 h_2\left[\frac{\partial}{\partial\xi}\left(\frac{U_\xi}{h_2}\right) + \frac{\partial}{\partial\eta}\left(\frac{U_\eta}{h_1}\right)\right] + \frac{1}{2}h_1 h_2\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2 + \left(\frac{\partial\omega}{\partial\eta}\right)^2\right] = \frac{\beta_1^2 h^2}{12} + (1+\nu)\alpha_t T_0 \tag{34}$$

where

$$h_1 = h_2 = 1/d\sqrt{\sinh^2\xi + \sin^2\eta}.$$

The boundary conditions for U_ξ and U_η are

$$U_\xi = 0 = U_\eta \text{ at } \xi = \xi_0 \tag{35}$$

Let us assume

$$U_\xi = \sum_{n=0}^{\infty} P_n(\xi)\cos 2n\eta$$

$$U_\eta = \sum_{n=1}^{\infty} G_n(\xi)\sin 2n\eta \tag{36}$$

subject to the conditions $P_n(\xi_0) = G_n(\xi_0) = 0$ (37)

The physical situation corresponding to the above assumed forms of U_ξ and U_η is that they satisfy the requisite conditions of immovable edges along the boundary, i.e., inplane displacements are restrained on the boundary.

U_ξ and U_η are also the displacements in elliptic coordinates corresponding to the in-plane displacements u and v in cartesian coordinates, and $u = 0$ along y-axis and $v = 0$ along x-axis.

Therefore $U_\xi = 0$ at $\eta = \pi/2$ and $U_\eta = 0$ at $\eta = 0$ are the corresponding conditions imposed on U_ξ and U_η. The above forms (36) clearly satisfy the required conditions.

Integrating (34) over the surface of the plate, one gets

$$\int_0^{2\pi}\int_0^{\xi_0}\left[\left(\frac{\partial \omega}{\partial \xi}\right)^2+\left(\frac{\partial \omega}{\partial \eta}\right)^2\right] d\xi d\eta - 2(1+\nu)\,\alpha_t \int_0^{2\pi}\int_0^{\xi_0} T_0(\xi,\eta)(\sinh^2\xi+\sin^2\eta)\,d\xi d\eta - \frac{d^2\beta_1^2 h^2}{6}\int_0^{2\pi}\int_0^{\xi_0}(\sinh^2\xi+\sin^2\eta)\,d\xi d\eta = 0 \tag{38}$$

Substituting the values of ω and T_0 given by (30) and (17) respectively, equation (38), after integration, becomes an equation for determining β_1^2.

NUMERICAL CALCULATION

As a particular case T_0 can be taken to be a constant, for $T_0 =$ constant is a solution of the differential equation (15).

For determining the deflection at a point one has to start from equation (38) with an assumed value of β_1^2 leading to a particular value of $\lambda = \dfrac{E\alpha_t f(h)}{D(1-\nu)}$

With these values of λ and β_1^2, deflection ω is to be calculated from (30). With the following data

$$\xi = 0,\ \eta = \pi/2,\ \xi_0 = 3,\ d = 2,\ h = 1,\ \beta_1^2 = 0{\cdot}01,\ \epsilon = 0{\cdot}03,\ \alpha_t T_0 = 1{\cdot}2 \times 10^{-3},$$

the central deflection is obtained as $\omega = 0{\cdot}0025$ (approx.).

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