LIFE TESTS WITH PERIODIC CHANGE IN LIFE DISTRIBUTION PARAMETERS (NORMAL DISTRIBUTION)

T. N. SRIVASTAVA & K. E. SREEDHARAN

Institute of Armament Technology, Poona

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In this paper two life testing procedures namely the progressively censored samples and Bartholomew's experiment have been discussed. The discussions are based on the assumption that the life of an item follows normal distribution such that the mean and standard deviation are different under two different conditions of usage of the item at regular intervals of time. The maximum likelihood estimates of the two parameters have been derived along with their variances.

Aroian[1], Srivastava[2,3] have studied life test experiments where the failure rate of an item changes, with time. The failure rate of an item may also change with the change in conditions of usage. They have assumed the life distribution to be negative exponential. Gajjar and Khatri[4] have discussed life test experiments where the life distributions assumed are Log Normal and Logistic. They assumed that the life distribution parameters undergo change at specified times. However, we can visualise situations where an item will be used under only two different conditions one after the other in a cycle.

In this paper it has been assumed that the life of an item follows normal distribution. However, the parameters are different under the two different conditions of usage at regular intervals of time. Two life test experiments namely the progressively censored samples envisaged by Cohen[5] and Bartholomew's[6] experiment have been considered.

MODEL

The probability density function of the random variable t representing the life of an item, having a normal distribution with parameters μ and σ (mean and standard deviation, respectively) is given by

$$f(t;\mu,\sigma)=(1/\sigma\sqrt{2\pi})\exp\left\{(-1/2)(t-\mu)^2/\sigma^2\right\};\ -\infty<t<\infty \tag{1}$$

For the situation under consideration in this paper the probability density function of the variable t representing the life of an item can be written as

$$f(t)=\begin{cases} a_{1,j}f_1(t); & (j-1)(T_1+T_2)<t\leqslant(j-1)(T_1+T_2)+T_1 \\ a_{2,j}f_2(t); & (j-1)(T_1+T_2)+T_1<t\leqslant j(T_1+T_2)\end{cases} \tag{2}$$

The corresponding distribution function of t is given by

$$1-F(t)=\begin{cases} a_{1,j}[1-F_1(t)]; & (j-1)(T_1+T_2)<t\leqslant(j-1)(T_1+T_2)+T_1 \\ a_{2,j}[1-F_2(t)]; & (j-1)(T_1+T_2)+T_1<t\leqslant j(T_1+T_2)\end{cases} \tag{3}$$

where

$$F_i(t)=\int_{-\infty}^{t}f_i(t)\,dt, \qquad i=1,2.$$

$$j=(1,2,\ldots\ldots\ldots\ldots)\text{ represents } j\text{th cycle}$$

and

$$a_{1,1}=\frac{1}{1-F(1,1)}$$

$$a_{2,1} = \frac{1}{1-F(1,1)} \quad \frac{1-F(1,1)_1}{1-F(1,2)_1}$$

$$\left.\begin{aligned} a_{1,j} &= \frac{1}{1-F(1,1)} \prod_{m=2}^{j} \frac{1-F(m-1,1)_1}{1-F(m-1,2)_1} \cdot \frac{1-F(m,2)}{1-F(m,1)} ; \\ a_{2,j} &= \frac{1}{1-F(1,1)} \cdot \frac{1-F(1,1)_1}{1-F(1,2)_1} \prod_{m=2}^{j} \frac{1-F(m,2)}{1-F(m,1)} \cdot \frac{1-F(m,1)_1}{1-F(m,2)_1} ; \end{aligned}\right\} j=2,3,\ldots$$

where

$$F(j,i) = F_i\left[(j-1)(T_1+T_2)\right]; \qquad i=1,2.$$

$$F(j,i)_1 = F_i\left[(j-1)(T_1+T_2)+T_1\right]; \qquad j=1,2,\ldots\ldots$$

It may be noted that time period of each cycle is $T_1 + T_2$; the mean and standard deviations are μ_1 and σ_1, respectively, during the first time interval T_1 and μ_2 and σ_2 during the second time interval T_2.

PROGRESSIVELY CENSORED SAMPLES

Life testing experiments, involving progressively censored samples, were envisaged by Cohen[5]. In such an experiment certain known number of items are placed on test to start with. However, because of the need of these items somewhere else some of these are removed from the experiment at some predetermined times. Such type of censoring has been referred to as Type I by Cohen.

In this section maximum likelihood estimates of the parameters μ_1, μ_2, σ_1 and σ_2 alongwith their asymptotic variances for the above experimental situation have been derived.

Let n items be placed on test. In the jth cycle, let n_{1j} be the number of items that failed by item T_1, n_{2j} be the number of items that failed between times T_1 and $T_1 + T_2$, r_{1j} be the number of items removed after time T_1 and r_{2j} be the number of items removed after time $T_1 + T_2$.

Let the experiment be finally terminated after kth cycle.

ESTIMATION OF PARAMETERS $\mu_1, \sigma_1, \mu_2,$ AND σ_2

The likelihood function of the sample arising as a result of the above experiment is given by

$$P(S) = \prod_{j=1}^{k} \prod_{i=1}^{2} \left[\left\{ \prod_{p=1}^{n_{ij}} f\,(t_{pi}{}^{j}) \right\} (a_{i,j})^{r_{ij}} \left\{ 1 - F_i\,(\overline{j-1}\;\overline{T_1+T_2} + T_1 + \overline{i-1}\;T_2) \right\}^{r_{ij}} \right]$$

$$= \prod_{j=1}^{k} \left[\prod_{i=1}^{2} (a_{i,j})^{n_{ij}+r_{ij}} (\sigma_i)^{-n_{ij}} \exp\left\{ -(1/2\sigma_i^2) \sum_{p=1}^{n_{ij}} (t_{pi}{}^{j} - \mu_i)^2 \right\} \right.$$

$$\left. \cdot \left\{ 1 - F(j,1)_1 \right\}^{r_{1j}} \left\{ 1 - F(j+1,2) \right\}^{r_{2j}} \right]$$

It may be noted that $\prod_{p=1}^{n_{ij}}$ and $\sum_{p=1}^{n_{ij}}$ refer to product and summation taken over n_{ij} items that failed in the ith ($i = 1, 2$) part of the jth cycle. $t_{pi}{}^{j}$ are the times at which the pth item ($p = 1, 2, \ldots\ldots, n_{ij}$) fail in the ith part ($i = 1,2$) of the jth cycle ($j = 1, 2, \ldots\ldots\ldots\ldots, k$).

Taking logarithm of the likelihood function, we get

$$L \text{ (say)} = \log P(S)$$

$$= \sum_{j=1}^{k} \Big[\sum_{i=1}^{2} \Big\{ (n_{ij} + r_{ij}) (\log a_{i,j} - \log \sigma_i) -$$

$$- (1/2\sigma_i^2) \sum_{p=1}^{n_{ij}} (t_{pi}{}^j - \mu_i)^2 + r_{1j} \log \Big\{ 1 - F(j, 1)_1 \Big\} + r_{2j} \log \Big\{ 1 - F(j+1, 2) \Big\} \Big]$$

$$= \sum_{j=2}^{k} \Big\{ (n_{1j} + r_{1j}) (\log a_{1,j} - \log \sigma_1) + (n_{2j} + r_{2j}) (\log a_{2,j} - \log \sigma_2) \Big\} +$$

$$+ \sum_{j=1}^{k} \Big[r_{1j} \log \Big\{ 1 - F(j, 1)_1 \Big\} + r_{2j} \log \Big\{ 1 - F(j+1, 2) \Big\} \Big] +$$

$$+ (n_{11} + r_{11}) (\log a_{1,1} - \log \sigma_1) + (n_{21} + r_{21}) (\log a_{2,1} - \log \sigma_2) -$$

$$- \sum_{j=1}^{k} \sum_{i=1}^{2} (1/2\ \sigma_i^2) \sum_{p=1}^{ij} (t_{pi}{}^j - \mu_i)^2$$

Differentiating L with respect to μ_1, σ_1, μ_2 and σ_2 and equating to zero, we get the maximum likelihood estimating equations, for these parameters

$$\sigma_1 \frac{\partial L}{\partial \mu_1} = -n\, Z(1, 1) + Z(1, 1)_1 \sum_{j=1}^{k} (n_{2j} + r_{2j}) +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \Big[(n_{1j} + r_{1j}) \Big\{ Z(m-1, 1)_1 - Z(m, 1) \Big\} +$$

$$+ (n_{2j} + r_{2j}) \Big\{ Z(m, 1)_1 - Z(m, 1) \Big\} \Big] +$$

$$+ \sum_{j=1}^{k} \Big\{ r_{1j}\ Z(j, 1)_1 + (1/\sigma_1) \sum_{p=1}^{n_{1j}} (t_{p1}{}^j - \mu_1) \Big\} = 0 \tag{4}$$

$$\sigma_1 \frac{\partial L}{\partial \sigma_1} = -n(1, 1)\, Z(1, 1) + (1, 1)_1\, Z(1, 1)_1 \sum_{j=1}^{k} (n_{2j} + r_{2j}) +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \Big[(n_{1j} + r_{1j}) \Big\{ (m-1, 1)_1\, Z(m-1, 1)_1 - (m, 1)\, Z(m, 1) \Big\} +$$

$$+ (n_{2j} + r_{2j}) \Big\{ (m, 1)_1\ Z(m, 1)_1 - (m, 1)\ Z(m, 1) \Big\} \Big] +$$

$$+ \sum_{j=1}^{k} \Big\{ r_{1j} (j, 1)_1\ Z(j, 1)_1 + (1/\sigma_1^2) \sum_{p=1}^{n_{1j}} (t_{p1}{}^j - \mu_1)^2 - n_{1j} - r_{1j} \Big\} = 0 \tag{5}$$

$$\sigma_2 \frac{\partial L}{\partial \mu_2} = - Z(1,2)_1 \sum_{j=1}^{k} (n_{2j} + r_{2j}) +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \Big[(n_{1j} + r_{1j}) \Big\{ Z(m,2) - Z(m-1,2)_1 \Big\} +$$

$$+ (n_{2j} + r_{2j}) \Big\{ Z(m,2) - Z(m,2)_1 \Big\} \Big] +$$

$$+ \sum_{j=1}^{k} \Big\{ r_{2j} Z(j+1,2) + (1/\sigma_2) \sum_{p=1}^{n_{2j}} (t_{p2}{}^{j} - \mu_2) \Big\} = 0$$

$$\sigma_2 \frac{\partial L}{\partial \sigma_2} = - (1,2)_1 Z(1,2)_1 \sum_{j=1}^{k} (n_{2j} + r_{2j}) +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \Big[(n_{1j} + r_{1j}) \Big\{ (m,2)\, Z(m,2) - (m-1,2)_1\, Z(m-1,2)_1 \Big\} +$$

$$+ (n_{2j} + r_{2j}) \Big\{ (m,2)\, Z(m,2) - (m,2)_1\, Z(m,2)_1 \Big\} \Big] +$$

$$+ \sum_{j=1}^{k} \Big\{ r_{2j}\, (j+1,2)\, Z(j+1,2) + (1/\sigma_2{}^2) \sum_{p=1}^{n2_j} (t_{p2}{}^{j} - \mu_2)^2 - n_{2j} - r_{2j} \Big\} = 0 \quad (7)$$

where

$$\left.\begin{aligned} (j,i) &= \Big\{ (j-1)(T_1+T_2) - \mu_i \Big\} \Big/ \sigma_i \\ (j,i)_1 &= \Big\{ (j-1)(T_1+T_2) + T_1 - \mu_i \Big\} \Big/ \sigma_i \\ Z(j,i) &= Z \Big\{ (j-1)(T_1+T_2) - \mu_i \Big\} \Big/ \sigma_i \\ Z(j,i)_1 &= Z \Big\{ (j-1)(T_1+T_2) + T_1 - \mu_i \Big\} \Big/ \sigma_i \end{aligned}\right\} \quad \begin{aligned} i &= 1, 2. \\ j &= 1, 2, \ldots\ldots k \end{aligned}$$

$Z(t)$'s being the hazard rate or reciprocal of the Mill's ratio, $\Big\{1 - F(t)\Big\} \Big/ f(t)$

where

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \text{ and } F(t) = \int_{-\infty}^{t} f(t)\, dt.$$

Equations (4) and (5) can be solved with the help of Newton-Raphson's iterative method to get estimates of μ_1 and σ_1. Similarly equations (6) and (7) can provide estimates of μ_2 and σ_2.

VARIANCES OF μ_1, σ_1, μ_2 AND σ_2

It can be easily verified that

$$E\left(-\frac{\partial^2 L}{\partial \mu_1 \partial \mu_2}\right) = E\left(-\frac{\partial^2 L}{\partial \mu_1 \partial \sigma_2}\right) = E\left(-\frac{\partial^2 L}{\partial \mu_2 \partial \sigma_1}\right) = E\left(-\frac{\partial^2 L}{\partial \sigma_1 \partial \sigma_2}\right) = 0$$

Hence the asymptotic variance-covariances of the maximum likelihood estimates of μ_1, σ_1, μ_2 and σ_2 are given by the following matrix

$$\begin{bmatrix} E\left(-\frac{\partial^2 L}{\partial \mu_1^2}\right) & E\left(-\frac{\partial^2 L}{\partial \mu_1 \partial \sigma_1}\right) & 0 & 0 \\ E\left(-\frac{\partial^2 L}{\partial \mu_1 \partial \sigma_1}\right) & E\left(-\frac{\partial^2 L}{\partial \sigma_1^2}\right) & 0 & 0 \\ 0 & 0 & E\left(-\frac{\partial^2 L}{\partial \mu_2^2}\right) & E\left(-\frac{\partial^2 L}{\partial \mu_2 \partial \sigma_2}\right) \\ 0 & 0 & E\left(-\frac{\partial^2 L}{\partial \mu_2 \partial \sigma_2}\right) & E\left(-\frac{\partial^2 L}{\partial \sigma_2^2}\right) \end{bmatrix}^{-1}$$

It can be easily verified that

$$V(\hat{\mu}_i) = \frac{L(\sigma_i, \sigma_i)}{D(\mu_i, \sigma_i)}$$

$$V(\hat{\sigma}_i) = \frac{L(\mu_i, \mu_i)}{D(\mu_i, \sigma_i)} \qquad \text{Where} \qquad i = 1, 2.$$

$$Cov(\hat{\mu}_i, \hat{\sigma}_i) = -\frac{L(\mu_i, \sigma_i)}{D(\mu_i, \sigma_i)}$$

where

$$L(\mu_i, \sigma_i) = E\left(-\frac{\partial^2 L}{\partial \mu_i \partial \sigma_i}\right)$$

$$D(\mu_i, \sigma_i) = E\left(-\frac{\partial^2 L}{\partial \mu_i^2}\right) E\left(-\frac{\partial^2 L}{\partial \sigma_i^2}\right) - \left\{E\left(-\frac{\partial^2 L}{\partial \mu_i \partial \sigma_i}\right)\right\}^2$$

and so on.

Thus the asymptotic variance-covariance matrix of the estimates can be completed with the help of the following equations

$$\sigma_1^2 E\left(-\frac{\partial^2 L}{\partial \mu_1^2}\right) = -nA(1, 1) + A(1, 1)_1 \sum_{j=1}^{k} \left\{E(n_{2j}) + r_{2j}\right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{E(n_{1j}) + r_{1j}\right\} \left\{A(m-1, 1)_1 - A(m, 1)\right\} + \right.$$

$$\left. + \left\{E(n_{2j}) + r_{2j}\right\} \left\{A(m, 1)_1 - A(m, 1)\right\}\right] + \sum_{j=1}^{k} \left\{r_{1j} A(j, 1)_1 + E(n_{1j})\right\}$$

$$\sigma_1^2\, E\left(-\frac{\partial^2 L}{\partial \mu_1\, \partial \sigma_1}\right) = -\, nB(1,1) + B(1,1)_1 \sum_{j=1}^{k} \left\{ E(n_{2j}) + r_{2j} \right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{ E(n_{1j}) + r_{1j} \right\} \left\{ B(m-1,1)_1 - B(m,1) \right\} + \right.$$

$$\left. + \left\{ E(n_{2j}) + r_{2j} \right\} \left\{ B(m,1)_1 - B(m,1) \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left[r_{1j}\, B(j,1)_1 + (2/\sigma_1)\, E(n_{1j})\, E \left\{ (t-\mu_1) \,|\, (j,1) < t < (j,1)_1 \right\} \right]$$

$$E\left(-\frac{\partial^2 L}{\partial \sigma_1^2}\right) = -\, n\, C(1,1) + C(1,1)_1 \sum_{j=1}^{k} \left\{ E(n_{2j}) + r_{2j} \right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{ E(n_{1j}) + r_{1j} \right\} \left\{ C(m-1,1)_1 - C(m,1) \right\} + \right.$$

$$\left. + \left\{ E(n_{2j}) + r_{2j} \right\} \left\{ C(m,1)_1 - C(m,1) \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left[r_{1j}\, C(j,1)_1 + (3/\sigma_1^2)\, E(n_{1j})\, E \left\{ (t-\mu_1)^2 \,|\, (j,1) < t < (j,1)_1 \right\} - \right.$$

$$\left. -\, E(n_{1j}) - r_{1j} \right]$$

$$\sigma_2^2\, E\left(-\frac{\partial^2 L}{\partial \mu_2^2}\right) = -\, A(1,2)_1 \sum_{j=1}^{k} \left\{ E(n_{2j}) + r_{2j} \right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{ E(n_{1j}) + r_{1j} \right\} \left\{ A(m,2) - A(m-1,2)_1 \right\} + \right.$$

$$\left. + \left\{ E(n_{2j}) + r_{2j} \right\} \left\{ A(m,2) - A(m,2)_1 \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left\{ r_{2j}\, A(j+1,2) + E(n_{2j}) \right\}$$

$$\sigma_2^2\, E\left(-\frac{\partial^2 L}{\partial \mu_2\, \partial \sigma_2}\right) = -\, B(1,2)_1 \sum_{j=1}^{k} \left\{ E(n_{2j}) + r_{2j} \right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{ E(n_{1j}) + r_{1j} \right\} \left\{ B(m,2) - B(m-1,2)_1 \right\} + \right.$$

$$+ \left\{ E(n_{2j}) + r_{2j} \right\} \left\{ B(m, 2) - B(m, 2)_1 \right\} + \sum_{j=1}^{k} \left[r_{2j} B(j+1, 2) + \right.$$

$$\left. + (2/\sigma_2) E(n_{2j}) E \left\{ (t - \mu_2) \mid (j, 2)_1 < t < (j+1, 2) \right\} \right]$$

$$\sigma_2^2 E \left(- \frac{\partial^2 L}{\partial \sigma_2^2} \right) = - C(1, 2)_1 \sum_{j=1}^{k} \left\{ E(n_{2j}) + r_{2j} \right\} +$$

$$+ \sum_{j=2}^{k} \sum_{m=2}^{j} \left[\left\{ E(n_{1j}) + r_{1j} \right\} \left\{ C(m, 2) - C(m-1, 2)_1 \right\} + \right.$$

$$\left. + \left\{ E(n_{2j}) + r_{2j} \right\} \left\{ C(m, 2) - C(m, 2)_1 \right\} \right] + \sum_{j=1}^{k} \left[r_{2j} C(j+1, 2) + \right.$$

$$+ (3/\sigma_2^2) E(n_{2j}) E \left\{ (t - \mu_2)^2 \mid (j, 2)_1 < t < (j+1, 2) \right\} -$$

$$\left. - E(n_{2j}) - r_{2j} \right]$$

where

$$A(.) = Z(.) \{ Z(.) - (.) \}$$
$$B(.) = Z(.) + (.) A(.)$$
$$C(.) = (.) \{ Z(.) + B(.) \}$$

$$E \left\{ (t - \mu_1) \mid (j, 1) < t < (j, 1)_1 \right\} = \frac{\frac{\sigma_1}{\sqrt{2\pi}} \left[\exp[-\tfrac{1}{2} \{ (j, 1) \}^2] - \exp[-\tfrac{1}{2} \{ (j, 1)_1 \}^2] \right]}{F(j, 1)_1 - F(j, 1)}$$

$$E \left\{ (t - \mu_2) \mid (j, 2)_1 < t < (j+1, 2) \right\} = \frac{\frac{\sigma_2}{\sqrt{2\pi}} \left[\exp[-\tfrac{1}{2} \{ (j, 2)_1 \}^2] - \exp[-\tfrac{1}{2} \{ (j+1, 2) \}^2] \right]}{F(j+1, 2) - F(j, 2)_1}$$

$$E \left\{ (t - \mu_1)^2 \mid (j, 1) < t < (j, 1)_1 \right\} = \int_{\frac{1}{2}\{(j, 1)\}^2}^{\frac{1}{2}\{(j, 1)_1\}^2} u^{1/2} e^{-u} du \Big/ \left\{ F(j, 1)_1 - F(j, 1) \right\}$$

$$E \left\{ (t - \mu_2)^2 \mid (j, 2)_1 < t < (j+1, 2) \right\} = \int_{\frac{1}{2}\{(j, 2)_1\}^2}^{\frac{1}{2}\{(j+1, 2)\}^2} u^{1/2} e^{-u} du \Big/ \left\{ F(j+1, 2) - F(j, 2)_1 \right\}$$

Further, it can be easily derived that

$$E(n_{11}) = n \, \frac{F(1, 1)_1 - F(1, 1)}{1 - F(1, 1)}$$

$$E(n_{21}) = \left\{ n - r_{11} - E(n_{11}) \right\} \left\{ \frac{F(2, 2) - F(1, 2)_1}{1 - F(1, 2)_1} \right\};$$

$$\left.\begin{aligned} E(n_{1j}) &= \left[n - \sum_{m=1}^{j-1} \left\{ r_{1m} + r_{2m} + E(n_{1m} + n_{2m}) \right\} \right] \left[\frac{F(j,1)_1 - F(j,1)}{1 - F(j,1)} \right]; \\ E(n_{2j}) &= \left[n - \sum_{m=1}^{j-1} \left\{ r_{1m} + r_{2m} + E(n_{1m} + n_{2m}) \right\} - r_{1j} - E(n_{1j}) \right] \times \\ &\quad \times \left[\frac{F(j+1,2) - F(j,2)_1}{1 - F(j,2)_1} \right]. \end{aligned} \right\} j = 2, 3 \ldots k$$

BARTHOLOMEW'S EXPERIMENT

Bartholomew[6] envisaged a life testing experiment in which all the items are placed on test at different times depending on their availability. A generalised form of such an experiment could be as follows.

Let a sample of N_1 items be placed on test initially and samples of N_j items be placed on test after a time $(j-1)(T_1 + T_2)$ elapses from the start of the experiment $(j = 2, \ldots\ldots k)$. Let the experiment be terminated after time $k(T_1 + T_2)$ elapses from the start of the experiment.

It may be noted that in its general form Bartholomew's experiment could be termed as "progressively added samples" corresponding to progressively censored samples introduced by Cohen.

In this section the maximum likelihood estimates of the parameters μ_1, μ_2, σ_1 and σ_2 along with their asymptotic variances for the above generalised experimental situation have been derived.

Let n be the number of items that failed during the experiment and r the number of items that survive. Among the r items that survive, let r_i be from the ith sample of N_i items $(i = 1, 2, \ldots\ldots k)$. Further let,

n_{1j} be the number of items that failed in the first part of the jth cycle, i.e. between $(j-1)(T_1 + T_2)$ and $(j-1)(T_1 + T_2) + T_1$,

n_{2j} be the number of items that failed in the second part of the jth cycle, i.e. between $(j-1)(T_1 + T_2) + T_1$ to $j(T_1 + T_2)$,

n_{1ji} be the number of items among the n_{1j} items from the ith sample,

n_{2ji} be the number of items among the n_{2j} items from the ith sample,

$$n_1 = \sum_{j=1}^{k} n_{1j} \; ; \quad \text{and} \quad n_2 = \sum_{j=1}^{k} n_{2j}$$

The following relations are then evident,

$$n_{1j} = \sum_{i=1}^{j} n_{1ji} \; ; \qquad n_{2j} = \sum_{i=1}^{j} n_{2ji}$$

$$\sum_{i=1}^{k} N_i = n_1 + n_2 + r = n + r = N \text{ (say)}.$$

ESTIMATION OF PARAMETERS μ_1, σ_1, μ_2 AND σ_2

The likelihood function of the sample arisen as a result of the above experiment is given by

$$P(S) = \prod_{j=1}^{k} (a_{2j})^{r_{k+1-j}} \left\{ 1 - F(j+1, 2) \right\}^{r_{k+1-j}} \times$$
$$\times \prod_{i=1}^{j} \left\{ a_{1,(j+1-i)} \right\}^{n_{1ji}} \cdot (\sigma_1)^{n_{1ji}} \times$$
$$\times \exp \left\{ (-1/2\,\sigma_1^2) \sum_{p=1}^{n_{1ji}} (t_{p^1}^{j+1-i} - \mu_1)^2 \right\} \times$$
$$\times \left\{ a_{2,(j+1-i)} \right\}^{n_{2ji}} (\sigma_2)^{n_{2ji}} \times$$
$$\times \exp \left\{ (-1/2\,\sigma_2^2) \sum_{p=1}^{n_{2ji}} (t_{p^2}^{j+1-i} - \mu_2)^2 \right\}$$

where $t_{p^1}^{j+1-i}$ is the time at which the pth item of ith sample failed in the first part of jth cycle and $t_{p^2}^{j+1-i}$ is the time at which the pth item of ith sample failed in the second part of jth cycle.

Taking logarithm of the likelihood function we get,

L (say) $= \log P$ (S)

$$= \sum_{j=1}^{k} \Bigg[\Bigg[r_{k+1-j} \left\{ \log a_{2,j} + \log (1 - F\,\overline{j+1, 2}) \right\} \Bigg] +$$
$$+ \sum_{i=1}^{j} \Bigg[n_{1ji} \left\{ \log a_{1,(j+1-i)} + \log \sigma_1 \right\} +$$
$$+ n_{2ji} \left\{ \log a_{2,(j+1-i)} + \log \sigma_2 \right\} -$$
$$- \left\{ (1/2\,\sigma_1^2) \sum_{p=1}^{n_{1ji}} (t_{p^1}^{j+1-i} - \mu_1)^2 + \right.$$
$$\left. + (1/2\,\sigma_2^2) \sum_{p=1}^{n_{2ji}} (t_{p^2}^{j+1-i} - \mu_2)^2 \right\} \Bigg] \Bigg]$$

It may be noted that $\prod_{p=1}^{n_{1ji}}$ and $\sum_{p=1}^{n_{1ji}}$ refer to product and summation taken over n_{1ji} items of the ith sample that failed in first part of the jth cycle while $\prod_{p=1}^{n_{2ji}}$ and $\sum_{p=1}^{n_{2ji}}$ refer to product and summation taken over n_{2ji} items of the ith sample that failed in second part of the jth cycle. Differentiating L with respect to μ_1, σ_1, μ_2 and σ_2 and equating the derivatives to zero, we get the maximum likelihood estimating equations as follows:

$$\sigma_1 \frac{\partial L}{\partial \mu_1} = -N\,Z(1,1) + (r + n_2)\,Z(1,1)_1 +$$

$$+ \sum_{j=2}^{k} r_{k+1-j} \left[\sum_{m=2}^{j} \left\{ Z(m,1)_1 - Z(m,1) \right\} \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[n_{1ji} \sum_{m=2}^{j+1-i} \left\{ Z(m-1,1)_1 - Z(m,1) \right\} + \right.$$

$$\left. + n_{2ji} \sum_{m=2}^{j+1-i} \left\{ Z(m,1)_1 - Z(m,1) \right\} \right] +$$

$$+ \frac{1}{\sigma_1} \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{p=1}^{n_{1ji}} (t_{p^1}^{j+1-i} - \mu_1) = 0$$

$$\sigma_1 \frac{\partial L}{\partial \sigma_1} = n_1 - N(1,1)\, Z(1,1) + (r + n_2)\, (1,1)_1\, Z(1,1)_1 +$$

$$+ \sum_{j=2}^{k} r_{k+1-j} \left[\sum_{m=2}^{j} \left\{ (m,1)_1\, Z(m,1)_1 - (m,1)\, Z(m,1) \right\} \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[n_{1ji} \sum_{m=2}^{j+1-i} \left\{ (m-1,1)_1\, Z(m-1,1)_1 - (m,1)\, Z(m,1) \right\} + \right.$$

$$\left. + n_{2ji} \sum_{m=2}^{j+1-i} \left\{ (m,1)_1\, Z(m,1)_1 - (m,1)\, Z(m,1) \right\} \right] +$$

$$+ \frac{1}{\sigma_1^2} \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{p=1}^{n_{1ji}} (t_{p^1}^{j+1-i} - \mu_1)^2 = 0$$

$$\sigma_2 \frac{\partial L}{\partial \mu_2} = -(n_2 + r)\, Z(1,2)_1 + \sum_{j=2}^{k} r_{k+1-j} \left[\sum_{m=2}^{j} \left\{ Z(m,2) - Z(m,2)_1 \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left[r_{k+1-j}\, Z(j+1,2) \right] + \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[n_{1ji} \sum_{m=2}^{j+1-i} \left\{ Z(m,2) - Z(m-1,2)_1 \right\} + \right.$$

$$\left. + n_{2ji} \sum_{m=2}^{j+1-i} \left\{ Z(m,2) - Z(m,2)_1 \right\} \right] + \frac{1}{\sigma_2} \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{p=1}^{n_{2ji}} (t_{p^2}^{j+1-i} - \mu_2) = 0$$

$$\sigma_2 \frac{\partial L}{\partial \sigma_2} = n_2 - (n_2 + r)\, (1,2)_1\, Z(1,2)_1 +$$

$$+ \sum_{j=2}^{k} r_{k+1-j} \left[\sum_{m=2}^{j} \left\{ (m,2)\, Z(m,2) - (m,2)_1\, Z(m,2)_1 \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left[r_{k+1-j} \, (j+1, 2) \; Z\,(j+1, 2) \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[n_{1ji} \sum_{m=2}^{j+1-i} \left\{ (m, 2)\, Z\,(m, 2) - (m-1, 2)_1 \, Z\,(m-1, 2)_1 \right\} + \right.$$

$$\left. + \; n_{2ji} \sum_{m=2}^{j+1-i} \left\{ (m, 2)\, Z\,(m, 2) - (m, 2)_1 \, Z\,(m, 2)_1 \right\} \right] +$$

$$+ \; \frac{1}{\sigma_2^{\,2}} \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{p=1}^{n_{2ji}} (t_{p^2}^{\;j+1-i} - \mu_2)^2 = 0$$

ASYMPTOTIC VARIANCES OF $\hat{\mu}_1$, $\hat{\sigma}_1$, $\hat{\mu}_2$ AND $\hat{\sigma}_2$

The following equations help in calculating the variance-covariance matrix of the estimates.

$$\sigma_1^{\,2} \, E \left(- \frac{\partial^2 L}{\partial \, \mu_1^{\,2}} \right) = - N A\,(1, 1) + \left\{ N - E\,(n_1) \right\} A\,(1, 1)_1 +$$

$$+ \sum_{j=2}^{k} E\,(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{ A\,(m, 1)_1 - A\,(m, 1) \right\} \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[E\,(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{ A\,(m-1, 1)_1 - A\,(m, 1) \right\} + \right.$$

$$\left. + \; E\,(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{ A\,(m, 1)_1 - A\,(m, 1) \right\} \right] + \left(\frac{1}{\sigma_1} \right) \sum_{j=1}^{k} \sum_{i=1}^{j} E\,(n_{1ji})$$

$$\sigma_1^{\,2} \, E \left(- \frac{\partial^2 L}{\partial \, \mu_1 \, \partial \sigma_1} \right) = - N B\,(1, 1) + \left\{ N - E\,(n_1) \right\} B\,(1, 1)_1 +$$

$$+ \sum_{j=2}^{k} E\,(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{ B\,(m, 1)_1 - B\,(m, 1) \right\} \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[E\,(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{ B\,(m-1, 1)_1 - B\,(m, 1) \right\} + \right.$$

$$\left. + \; E\,(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{ B\,(m, 1)_1 - B\,(m, 1) \right\} \right] +$$

$$+ \left(\frac{2}{\sigma_1} \right) \sum_{j=1}^{k} \sum_{i=1}^{j} E\,(n_{1ji}) \, E \left\{ (t - \mu_1) \middle| \, (j - i + 1, 1) < t < (j - \right.$$

$$\left. - \, i + 1, 1)_1 \right\}$$

$$\sigma_1^2 E\left(-\frac{\partial^2 L}{\partial \sigma_1^2}\right) = n_1 - N C(1,1) + \left\{N - E(n_1)\right\} C(1,1)_1 + $$
$$+ \sum_{j=2}^{k} E(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{C(m,1)_1 - C(m,1)\right\}\right] +$$
$$+ \sum_{j=2}^{k}\sum_{i=1}^{j-1} \left[E(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{C(m-1,1)_1 - C(m,1)\right\} +\right.$$
$$\left.+ E(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{C(m,1)_1 - C(m,1)\right\}\right] +$$
$$+ \left(\frac{3}{\sigma_1^2}\right) \sum_{j=1}^{k}\sum_{i=1}^{j} E(n_{1ji})\, E\left\{(t-\mu_1)^2 \,|\, (j-i+1,1) < t < (j-\right.$$
$$\left. - i+1,1)_1\right\}$$

$$\sigma_2^2 E\left(-\frac{\partial^2 L}{\partial \mu_2^2}\right) = -\left\{N - E(n_1)\right\} A(1,2)_1 +$$
$$+ \sum_{j=2}^{k} E(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{A(m,2) - A(m,2)_1\right\}\right] +$$
$$+ \sum_{j=1}^{k} \left[E(r_{k+1-j})\, A(j+1,2)\right] +$$
$$+ \sum_{j=2}^{k}\sum_{i=1}^{j-1} \left[E(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{A(m,2) - A(m-1,2)_1\right\} +\right.$$
$$\left.+ E(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{A(m,2) - A(m,2)_1\right\}\right] + \left(\frac{1}{\sigma_2}\right) \sum_{j=1}^{k}\sum_{i=1}^{j} E(n_{2ji})$$

$$\sigma_2^2 E\left(-\frac{\partial^2 L}{\partial \mu_2\, \partial \sigma_2}\right) = -\left\{N - E(n_1)\right\} B(1,2)_1 +$$
$$+ \sum_{j=2}^{k} E(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{B(m,2) - B(m,2)_1\right\}\right] + \sum_{j=1}^{k} \left[E(r_{k+1-j}) \cdot\right.$$
$$\left.\cdot\, B(j+1,2)\right] + \sum_{j=2}^{k}\sum_{i=1}^{j-1} \left[E(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{B(m,2) - B(m-1,2)_1\right\} +\right.$$
$$\left.+ E(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{B(m,2) - B(m,2)_1\right\}\right] +$$

$$+ \left(\frac{2}{\sigma_2}\right) \sum_{j=1}^{k} \sum_{i=1}^{j} E(n_{2ji})\, E\left\{ (t-\mu_2) \mid (j-i+1,2)_1 < t < (j-i+1,2) \right\}$$

$$\sigma_2^2 E\left(-\frac{\partial^2 L}{\partial \sigma_2^2}\right) = n_2 - \left\{ N - E(n_1) \right\} C(1,2)_1 +$$

$$+ \sum_{j=2}^{k} E(r_{k+1-j}) \left[\sum_{m=2}^{j} \left\{ C(m,2) - C(m,2)_1 \right\} \right] +$$

$$+ \sum_{j=1}^{k} \left[E(r_{k+1-j})\, C(j+1,2) \right] +$$

$$+ \sum_{j=2}^{k} \sum_{i=1}^{j-1} \left[E(n_{1ji}) \sum_{m=2}^{j+1-i} \left\{ C(m,2) - C(m-1,2)_1 \right\} + \right.$$

$$\left. + E(n_{2ji}) \sum_{m=2}^{j+1-i} \left\{ C(m,2) - C(m,2)_1 \right\} \right] +$$

$$+ \left(\frac{3}{\sigma_2^2}\right) \sum_{j=1}^{k} \sum_{i=1}^{j} (n_{2ji})\, E\left\{ (t-\mu_2)^2 \mid (j-i+1,2)_1 < t < (j-i+1,2) \right\}$$

Further, it can be easily derived that

$$E(n_{11}) = N_1 \frac{F(1,1)_1 - F(1,1)}{1 - F(1,1)}$$

$$E(n_{21}) = \left\{ N_1 - E(n_{11}) \right\} \left\{ \frac{F(2,2) - F(1,2)_1}{1 - F(1,2)_1} \right\}$$

$$E(n_{1ji}) = \left\{ N_i - \sum_{q=1}^{2} \sum_{m=i}^{j=1} E(N_{qmi}) \right\} \left\{ \frac{F(j,1)_1 - F(j,1)}{1 - F(j,1)} \right\}; \text{ where } i = 1, 2 \ldots, j-1$$

$$= N_j \left\{ \frac{F(j,1)_1 - F(j,1)}{1 - F(j,1)} \right\}; \text{ where } i = j \text{ and } j = 2, 3, \ldots k.$$

$$E(n_{2ji}) = \left\{ N_i - E(n_{1ji}) \right\} \left\{ \frac{F(j+1,2) - F(j,2)_1}{1 - F(j,2)_1} \right\}; \text{ where } i = j$$

$$= \left\{ N_i - \sum_{q=1}^{2} \sum_{m=i}^{j-i} E(n_{qmi}) - E(n_{1ji}) \right\} \left\{ \frac{F(j+1,2) - F(j,2)_1}{1 - F(j,2)_1} \right\};$$

where $i = 1, 2, \ldots, j-1$

and

$$E(r_j) = N_j - \sum_{p=j}^{k} E(n_{1pj} + n_{2pj}); \text{ where } j = 1, 2 \ldots, k.$$

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