A SHOCK WAVE ATTACHED TO A POINTED OBSTACLE IN A STEADY FLOW OF A DISSOCIATING GAS

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A pointed obstacle is assumed symmetrically placed with respect to a uniform supersonic flow ahead of it. It is assumed that an oblique shock wave attached to the leading edge of the obstacle appears from the vertex so that the flow after the shock is along the surface of the obstacle. In this paper a relation between the curvature of an attached shock wave and that of a stream line is discovered. It is concluded that in a steady flow of an ideal dissociating gas, the stream lines at the rear of a straight attached shock wave are necessarily curved lines, whereas in an ordinary gas flow only a straight line flow is possible behind a straight shock.

The sudden and abrupt changes in the medium due to the appearance of discontinuities created by the motion of supersonic jet-aircrafts cause electronic vibrations effecting the balance, stability and control of the jet-aircrafts. Several researches have been carried out in this field to overcome or to minimize the effects of these undesirable vibrations.

The non-equilibrium flow past a blunt body have been discussed by Freeman[1], Gibson and Marrone[2], Lick[3] and many others. Sedney and Gerber[4] have treated the problem of determination of shock curvature and flow variable gradients at the tip of a pointed body in a non-equilibrium flow. Shankar[5] studied the singular surfaces of order one in non-equilibrium dissociative gas dynamics. Recently Ram and Sharma[6] treated the problem of regular reflection of an oblique shock in a plane flow of an ideal dissociating gas in the presence of transverse magnetic field.

In this paper we have discussed an interesting problem of the occurance of an attached shock wave in an ideal dissociating gas at the nose of a pointed obstacle. We have obtained an interesting relation under which the effects of compression due to the shock can be minimized. We have also established an interesting relation between the curvature of the attached shock wave and that of a stream line.

In case of hypersonic flights the kinetic energy of the re-entering craft is dissipated by the atmospheric gas through shock compression and viscous heating. The air molecules after absorbing this kinetic energy may go through a change of chemical composition. For the sake of simplicity a diatomic gas mixture is taken and each component of the gas mixture is assumed to be thermally perfect. The temperature range is taken to be $1000^\circ\ K \sim 7000^\circ\ K$ so that the only chemical reaction involved is that of dissociation and thus electronic excitation and ionization are neglected. The radiation heat loss from the mixture and the molecular transport effects leading to viscosity, diffusion and heat conduction are also neglected. In a dissociating diatomic gas the state of reacting mixture is uniquely described by the independent parameter such as the pressure P, the temperature T and the atom mass fraction α. In a frozen flow the atom mass fraction remain constant.

BASIC EQUATION

A simple dissociating gas is defined as a mixture resulting from a dissociation reaction in a symmetrical diatomic gas A_2, each A_2 molecule being made up from $2A_1$ atoms.

The reaction is

$$A_2 + X \underset{K_r}{\overset{K_f}{\rightleftharpoons}} 2A_1 + X ,$$

where the species X can be either A_2 or A_1. K_f and K_r are the reaction rate constant for forward and reverse reactions. Assuming the gas to be Lighthill's[7] dissociating one, the equation of continuity for the atom mass species for the steady flow can be written[8]

$$\bar{u}.\nabla \alpha = 4\rho K_r \left\{ \rho_d (1-\alpha) \exp \left(-\frac{T_d}{T}\right) - \rho\alpha^2 \right\} \left(\frac{D}{RT_d}\right)^2 (1+\alpha), \tag{1}$$

where ∇ is the two-dimensional nabla operator in the (x_1, x_2) plane. ρ and $\bar{u}$ are respectively the density and velocity vector of the gas mixture for two-dimensional flow under consideration. D, α, R, ρ_d and T_d are respectively the dissociation energy for unit mass, atom mass fraction, the gas constant for A_2, the characteristic density and characteristic temperature for dissociation.

The equations governing the two-dimensional flow under consideration are[8] :

$$\rho(\nabla \cdot \bar{u}) + \bar{u}\cdot\nabla \rho = 0, \tag{2}$$

$$\rho(\bar{u}\cdot\nabla)\bar{u} + \nabla P = 0, \tag{3}$$

$$\rho(\bar{u}\cdot\nabla)h + \bar{u}\cdot\nabla P = 0, \tag{4}$$

where the specific enthalpy h of the ideal dissociating gas is given by

$$h = \frac{P}{\rho}(4+\alpha)(1+\alpha)^{-1} + \alpha D. \tag{5}$$

In view of (1) and (5), the equation (4) can be written as

$$\bar{u}\cdot\nabla P + \rho a^2{}_f \nabla\cdot\bar{u} - \frac{4}{3}\rho k_r \left(\frac{D}{RT_d}\right)\left\{\rho_d(1-\alpha)\exp\left(-\frac{T_d}{T}\right) - \rho\alpha^2\right\} \times$$

$$\times \left\{3P - \rho D(1+\alpha)^2\right\}, \tag{6}$$

where a_f A is the frozen speed of sound given by

$$a_f^2 = P(4+\alpha)/3\rho.$$

The variation of ρ_d over the temperature range $1000^0 K \backsim 7000^\circ K$ is very slight. Hence for practical purpose, we may regard ρ_d as a constant.

SHOCK WAVE ATTACHED TO THE LEADING EDGE OF A POINTED OBSTACLE

Let us consider a plane uniform flow of an ideal dissociating gas and suppose that a pointed obstacle with its vertex at V is placed symmetrically with respect to a uniform flow infront of it. Let θ be the angle which the tangent to the obstacle at V makes with the x_1-axis in the direction of flow infront and x_2-axis is perpendicular to this direction in the plane of motion. Let ϕ be the angle which the tangent to the shock makes with the direction of flow infront. When the flow reaches the obstacle at V, it will deflect an angle θ. Now if the angle θ is less than the corresponding maximum angle of deflection across an oblique shock, the flow conditions may be represented by an oblique shock attached to the leading edge of the obstacle so that the flow after the shock is along the surface of the obstacle[9].

Let the shock-curve in a two-dimensional steady flow be given by

$$\bar{r} = \bar{r}(s), \tag{7}$$

where $\bar{r} = (x_1, x_2)$ is the position vector of a point P on the shock and the parameter s measures the arc-distance along the shock-curve. If $\bar{t}$ and $\bar{n}$ respectively denote the unit tangent and unit normal vectors to the shock at the point V, then

$$\partial_s \bar{r} = t, \quad \partial_s \bar{t} = k\bar{n}, \quad \partial_s \bar{n} = -k\bar{t}, \tag{8}$$

where K is the curvature of attached shock wave at the point V. We define the propagation of shock wave normal to itself. Let jump in any quantity Z across the shock be denoted by $(Z) = Z\text{-}Z_1$ where Z and Z_1 are values of Z just behind and just infront of the shock respectively. The geometrical compatibility condition of first order for the study of discontinuities in the continuum mechanics is[10]

$$[\nabla Z] = [\bar{n} \cdot \nabla Z]^{-} + (\bar{t} \cdot \nabla [Z])\,\bar{t}\,. \tag{9}$$

The fundamental system of equations for the discontinuity in an ideal dissociating gas flow can be written as[11]

$$\left.\begin{aligned} &[h + \tfrac{1}{2}u^2] = 0\,, \\ &[P] + \rho_1 u_{1n}[u_n] = 0\,, \\ &[\rho u_n] = 0\,, \end{aligned}\right\} \tag{10}$$

where $u_n = \bar{u}\cdot\bar{n}$. If we define the compression strength δ of the shock by relation $\delta = [\rho]/\rho_1$, we can deduce the following jump conditions from equation (10).

$$\begin{aligned} &[\bar{u}] = -\,\delta\,(1+\delta)^{-1}\,u_{1n}\,n\,; \\ &[P] = \delta\,(1+\delta)^{-1}\,\rho_1\,u^2_{1n}\,, \end{aligned} \tag{11}$$

δ is given by the quadratic equation

$$L_0 + L_1\,\delta + L_2\,\delta^2 = 0$$

where

$$L_0 = \pi\rho_1(1+\alpha_1)\left\{1+\alpha_1+[\alpha] - \frac{3P_1}{\rho_1(1+\alpha_1)}\right\}[\alpha]\,,$$

$$L_1 = 24\pi\rho_1(1+\alpha_1)\,u^2_{1n} - 8\pi P_1\{4+(5+\alpha_1)\alpha_1\} + 8\pi\,\times$$
$$\times\,\{2\rho_1 D(1+\alpha_1)(1+\alpha_1+[\alpha]) - P_1(7+\alpha_1)\}[\alpha]\,,$$

$$L_2 = 4\pi\rho_1(1+\alpha_1)(1+\alpha_1+[\alpha])(2D[\alpha] - u^2_{1n}) - 8\pi P_1\,\times$$
$$\times\,\{4+(5+\alpha_1)\alpha_1+(4+\alpha_1)[\alpha]\}\,.$$

In case of an attached shock, it can be shown that the strength of the shock depends upon the wave angle ϕ, the angle which the free stream makes the tangent to the shock. In view of (11) and from with the Fig. 1, we have

$$\frac{u_n}{u_{1n}} = \cot\phi\,\tan(\phi-\theta) = (1+\delta)^{-1},$$

hence, we get

$$\delta = \sin\theta\,\sec\phi\,\operatorname{cosec}(\phi-\theta)\,. \tag{12}$$

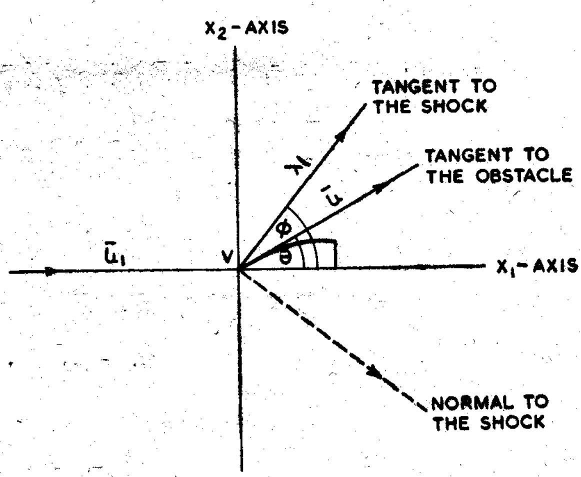

Fig. 1—Attached shock wave past a pointed obstacle.

Incidently from (12), we conclude the following important results :

(*i*) The physical appearance of an attached shock wave is possible only when ϕ satisfies the inequality

$\theta < \phi < \frac{\pi}{2}$.

(*ii*) In order to reduce the compression strength of the shock to the minimum the semi-vertex angle θ of the pointed obstacle should satisfy the condition

$$\phi = \frac{1}{2}\left(\theta + \frac{\pi}{2}\right).$$

The importance of these results can be very much realised in practical aeronautical problems. In fact the strong shock waves may cause sudden change in the aerodynamic behaviour of high speed Jet-aircrafts which effects their balance, stability and control by producing undesirable vibrations. The appearance of an attached shock wave is frequently observed by Jet-aircraft pilots [12]. The wave angle ϕ can be measured by optical methods. Thus by employing observational techniques and modelling the Jet-aircraft accordingly the compressions in the gas due to an attached shock wave can be minimized. The variation δ along the shock curve is given by

$$(\bar{t} \cdot \nabla)\, \delta = -\zeta k \tag{13}$$

where

$$\zeta = \sin\theta \cos(2\phi - \theta)/(\cos\phi \sin(\phi - \theta))^2 .$$

Differentiating (11) along the shock curve under the assumption of uniform flow infront of the shock and using (8) and (13) we obtain

$$\bar{t} \cdot \nabla P = -2\delta(1+\delta)^{-1} k\rho_1 u_{1n} u_{1t} - (1+\delta)^{-2} \zeta k \rho_1 u^2_{1n} , \tag{14}$$

$$(\bar{t} \cdot \nabla)\, \bar{u} = \delta k (1+\delta)^{-1} u_{1n} \bar{t} + k(1+\delta)^{-2} \{\delta(1+\delta) u_{1t} + \zeta u_{1n}\} \bar{n} , \tag{15}$$

where $u_{1t} = \bar{u}_1 \cdot \bar{t}$ and $u_{1n} = \bar{u}_1 \cdot \bar{n}$. By virtue of (9), the equations (1), (3) and (6) can be transformed into the following forms :

$$\bar{u}_n (\bar{n} \cdot \nabla)\rho + \bar{u}_t (\bar{t} \cdot \nabla)\rho + \rho n \cdot (n \cdot \nabla)\bar{u} + \rho \bar{t} \cdot (\bar{t} \cdot \nabla)\bar{u} = 0 , \tag{16}$$

$$\rho \bar{u}_n (\bar{n} \cdot \nabla)\bar{u} + \rho \bar{u}_t (\bar{t} \cdot \nabla)\bar{u} + \bar{n}(\bar{n} \cdot \nabla P) + \bar{t}(\bar{t} \cdot \nabla P) = 0 , \tag{17}$$

$$\left.\begin{aligned} &\bar{u}_n(\bar{n} \cdot \nabla P) + \bar{u}_t(\bar{t} \cdot \nabla P) + \rho a_f^2 \{ n \cdot (\bar{n} \cdot \nabla)\bar{u} + \bar{t} \cdot (\bar{t} \cdot \nabla)\bar{u} \} = \\ &= 4\rho K_r \left(\frac{D}{RT_d}\right)^2 \left\{ 3P - \rho D(1+\alpha)^2 \right\} \left\{ \rho_d (1-\alpha) \exp\left(-\frac{T_d}{T}\right) - \rho\alpha^2 \right\} . \end{aligned}\right\} \tag{18}$$

Taking dot product of (17) with $\bar{n}$ and using (18), we get

$$\begin{aligned} \bar{n} \cdot (\bar{n} \cdot \nabla)\bar{u} = \frac{1}{\rho} \{ &\bar{u}_t (\bar{t} \cdot \nabla P) + \rho a^2_f \, \bar{t} \cdot (\bar{t} \cdot \nabla)\bar{u} - \\ &- \rho \bar{u}_n \bar{u}_t n \cdot (\bar{t} \cdot \nabla)\bar{u} + \eta \} (u^2_n - a^2_f)^{-1} , \end{aligned} \tag{19}$$

where

$$\eta = 4\rho K_r \left(\frac{D}{RT_d}\right) \left\{ 3P - \rho D(1+\alpha)^2 \right\} \left\{ \rho_d (1-\alpha) \exp\left(\frac{-T_d}{T}\right) - \rho\alpha^2 \right\} .$$

Taking dot product of (17) with $\bar{t}$, we get

$$\rho \bar{u}_n \bar{t} \cdot (\bar{n} \cdot \nabla) \bar{u} = -\{\rho \bar{u}_t \bar{t} \cdot (\bar{t} \cdot \nabla) \bar{u} + (\bar{t} \cdot \nabla) P\}. \tag{20}$$

The curvature K of a stream line is given by

$$V^3 K = - e_{ik} u_k u_j u_{i,j}$$

where

$$V^2 = u_i \cdot u_i, \; e_{11} = e_{22} = 0, \; e_{12} = - e_{21} = 1.$$

Range of any dummy index say (i) is taken 1, 2 and a comma followed by an index say (j) denotes partial differentiation with respect to the corresponding coordinate x_j.

The curvature K at the rear of the shock at V is thus given by the expression.

$$K V^3 = \{ \bar{u}_n \bar{u}_t \bar{n} \cdot (\bar{n} \cdot \nabla) \bar{u} - \bar{u}^2{}_n \cdot \bar{t} (\bar{n} \cdot \nabla) \bar{u} + u_t^2 \bar{n} \cdot (t \cdot \nabla) \bar{u} - $$

$$- u_t u_n \bar{t} \cdot (\bar{t} \cdot \nabla) \bar{u} \}. \tag{21}$$

Substituting from (14), (15), (19) and (20), the equation (21) can be written in the form

$$K - 2\eta\xi \cos\phi = \xi k \; \bar{u}_1 \cos^2 \phi \; \{ \; \delta \; (\rho_1 a_f^2 + \rho M a_f^2 \cot \phi) + $$

$$+ (\rho\zeta M a^2{}_f - \frac{3}{2} \delta \rho_1 \bar{u}^2{}_1 \sin 2\phi) (1 + \delta)^{-1} - (1 + \delta)^{-2} .$$

$$\cdot (2\rho M \delta a^2{}_f + \zeta \rho_1 \bar{u}_1 \sin \phi + \frac{1}{2} \zeta \rho_1 \bar{u}_1^2 \sin 2\phi) \tan \phi - $$

$$- \rho M \zeta (1 + \delta)^{-3} a^2{}_f \}, \tag{22}$$

where

$$\xi = \frac{u_1^2 \sin \phi}{\rho M \, a^2{}_f (1 + \delta) V^3} \text{ and } M = \left(\frac{u_n^2}{a_f^2} - 1 \right).$$

The relation (22) provides a relation between the curvature of an attached shock wave and the curvature of a stream line at the rear of the shock at V.

Incidently from (22) we conclude that in a steady flow of an ideal dissociating gas the stream lines at the rear of a straight shock can never be straight whereas in absence of dissociation only a straight line flow is possible behind a straight shock wave.

Now, if the pointed obstacle is taken to be a wedge of semi-vertex angle θ with its vertex at V then the curvature of the stream line at V becomes zero and consequently the relation (22) determines the curvature of the shock wave at V. Hence, we further conclude that the attached shock wave in a uniform two-dimensional flow past a wedge becomes curved on account of dissociation.

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