THE STABILITY OF A VISCOUS FLOW BETWEEN TWO CONCENTRIC ROTATING POROUS CYLINDERS WITH AN AXIAL FLOW

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The combined effect of a radial and axial flow on the stability of a viscous flow between two concentric rotating cylinders has been examined numerically when the gap between the cylinders is assumed to be narrow. Numerical results show that the radial velocity induced by the suction of fluid from the outer cylinder tends to destabilize the flow whereas injection has a reverse effect.

The classical Taylor problem of the stability of Couette flow between two rotating cylinders in the presence of an axial flow has been studied theoretically by applying different approximate methods by Chandrasekhar[2,3], Di Prima[4], and Krueger & Di Prima[7]. In all these studies, the spacing between the cylindrical surfaces is assumed to be small as compared to the mean radius. Chandrasekhar[2] solved the problem by averaging the axial flow and the angular velocity distribution in the annulus. His results obtained by applying Fourier expansion technique for case of $\mu > 0$ (where μ is the angular velocity ratio of the cylinders) only, have been found to be fairly accurate by subsequent works. Di Prima[4] has studied the same problem by applying Galerkin method. By not averaging the axial flow, he was able to show that for smaller values (up to 30) of Reynolds number R associated with the axial velocity, it is quite reasonable to replace the parabolic profile of axial flow by its mean value Krueger & Di Prima[7] have extended this problem for the case of counter rotating cylinders i.e. when $\mu < 0$. Experimental studies of this problem have been made by Kaye & Elgar[6], Donnelly & Fultz[5], and Snyder[8]. The experimental results are in general in agreement with the theoretical results.

In the present work we have studied the effect of suction (or injection) on the stability of flow between two rotating porous cylinders with and without the presence of axial flow. Our study is restricted to the case of small spacing between the cylinders and for $\mu \geqslant 0$. We have averaged the axial flow and not the angular velocity distribution. The resulting sixth order differential equation with variable coefficients has been solved numerically by a numerical technique developed by Sparow, Munro & Jonsson (1964). Earlier Bahl[1] examined the effect of suction without any axial flow and solved the problem analytically by using Chandrasekhar's Fourier expansion technique.

EQUATIONS OF THE PROBLEM

The steady state solutions of the basic equations of motion and continuity governing an axisymmetric flow between two porous rotating cylinders with radial and axial flows give the velocity distribution in cylindrical coordinates (r, θ, z) as

$$u_r = \frac{R_1 u_1}{r} = U(r) \tag{1}$$

$$u_\theta = A r^{\lambda+1} + B/r = V(r) \tag{2}$$

$$u_z = \frac{-1}{2(2-\lambda)\nu\rho}\left(\frac{\partial P}{\partial z}\right)_0 \left[R^2{}_1 - r^2 - \frac{(R_2{}^2 - R_1{}^2)\{1-(r/R_1)^\lambda\}}{\left(1+\frac{d}{R_1}\right)^\lambda - 1} \right]$$

$$= W(r) \tag{3}$$

where

$$A = -\Omega_1 \eta^2 \frac{1 - \dfrac{\mu}{\eta^2}}{R_2^{\lambda}\,(1 - \eta^{\lambda+2})}$$

$$B = \Omega_1 R_1^2 \frac{(1 - \mu\eta^{\lambda})}{1 - \eta^{\lambda+2}}$$

$$\lambda = \frac{R_1 u_1}{\nu}\ ,\ \mu = \Omega_2/\Omega_1\ ,\ \eta = R_1/R_2,\ d = (R_2 - R_1)$$

$(\partial P/\partial z)_0$ is the constant axial pressure gradient and $W(r)$ vanishes at $r = R_1$ and R_2. In the above equations $(u_r,\ u_\theta,\ u_z)$ are the components of velocity in $(r,\ \theta,\ z)$ directions, R_1, Ω_1 are the radius and angular velocity of the inner cylinder and R_2, Ω_2 are the corresponding quantities of the outer cylinder, ρ the density and ν is the Kinematic viscosity.

Perturbation Equations

Now we superimpose small axisymmetric perturbations on the above stated steady state motion. Let the new disturbed velocity components be given as

$$\left.\begin{aligned} u_r &= U(r) + u(r)\, e^{i(pt + kz)} \\ u_\theta &= V(r) + v(r)\, e^{i(pt + kz)} \\ u_z &= W(r) + w(r)\, e^{i(pt + kz)} \\ \bar{w} &= \bar{w}(r)\, e^{i(pt + kz)} \end{aligned}\right\} \tag{4}$$

where

$$\bar{w} = \frac{\delta P}{\rho}\ ;$$ p is a constant and k is the wave number.

Substituting these in the basic equations of motion and continuity and neglecting the quadratic terms in the disturbances, we get the following equations

$$\nu\left[D D_* - k^2 - \frac{i\,(p + kW)}{\nu} \right] u - u\, DU - UD\, u + \frac{2V}{r}\, v = Dw \tag{5}$$

$$\nu\left[D D_* - k^2 - \frac{i\,(p + kW)}{\nu} \right] v - (D_* V)\, u - (D_* v)\, U = 0 \tag{6}$$

$$\nu\left[D D_* - k^2 - \frac{i\,(p + kW)}{\nu} \right] w - UDw - u\, DW = ik\bar{w} \tag{7}$$

$$D_* u = i\, k\, w \tag{8}$$

where

$$D = \frac{d}{dr} \quad \text{and} \quad D_* = \frac{d}{dr} + \frac{1}{r}$$

Narrow Gap Approximation

Applying narrow gap approximation by taking $d = (R_2 - R_1)$ to be small as compared with the mean radius $\dfrac{R_1 + R_2}{2}$, we can replace D_* by D. Also we get

$$\frac{V}{r} = Ar^{\lambda} + B/r^2 = \Omega_1 \left[1 - (1 - \mu)\, \zeta \right]$$

$$\frac{Wd}{\nu} = 6R\zeta\,(1 - \zeta)$$

where

$$R = \frac{V_m d}{\nu}, \quad V_m = \frac{-d^2}{12\rho\nu} \left(\frac{\partial P}{\partial z}\right)_0$$

and

$$\zeta = (r - R_1)/d$$

R is the Reynolds number corresponding to the mean axial flow V_m.

Now eliminating w & $\bar{w}$ from (5) to (8), non-dimensionalising the resulting equations by measuring distances in terms of d and transforming u to $\dfrac{2\Omega_1 d^2 a^2}{\nu} u$, we get the following two equations in the light of narrow gap approximation

$$\left[(D^2 - a^2) - i\left\{\sigma + 6Ra\,\zeta(1-\zeta)\right\}\right](D^2 - a^2)\,u - 12i\,Rau$$
$$= \left[1 - (1-\mu)\,\zeta\right]v \tag{9}$$

$$\left[(D^2 - a^2) - i\left\{\sigma + 6Ra\,\zeta(1-\zeta)\right\}\right]v$$
$$= -T(1 + \lambda/2)\,a^2(1 - k_1 + k_1\zeta)\,u \tag{10}$$

where

$D = \dfrac{d}{d\zeta}$, $a = kd$, $\sigma = \dfrac{pd^2}{\nu}$, $T = \dfrac{-4AR_2^{\lambda}\,\Omega_1 d^4}{\nu^2}$ is the Taylor number and $k_1 = \dfrac{\lambda d}{R_2}$.

Replacing the axial flow by its average value and eliminating v from (9) and (10) we get the following equation in u.

$$D^6u - 2\alpha\phi\,D^5u + (2\alpha^2\phi^2 - 3a^2 - 2iG)\,D^4u +$$
$$+(4a^2\alpha\phi + 2G\alpha\phi i)\,D^3u + \Big[-4a^2\alpha^2\phi^2 + 3a^4 -$$
$$-G^2 + i\,(4a^2G - H + 2\alpha^2G\phi^2)\Big]D^2u -$$
$$-(2\alpha a^4\phi + iC_1\alpha\phi)\,Du + \Big[(2\alpha^2 a^4\phi^2 + C_2) +$$
$$+ i\,(\alpha^2\phi^2C_1 + C_3) + \frac{Ta^2}{\phi}(1 - k_1 + k_1\zeta)(1 + \lambda/2)\Big]u = 0 \tag{11}$$

where

$$\alpha = (\mu - 1)$$
$$\phi = 1/(1 + \alpha\zeta)$$
$$G = \sigma + Ra, \quad H = 12Ra$$
$$C_1 = 2(a^2G - 2H)$$
$$C_2 = (a^2G - a^6 - HG)$$

and $C_3 = a^2H - 2a^4G$.

Boundary Conditions

The requirement that all the disturbances in velocity components vanish at the bounding surfaces $\zeta = 0$ and $\zeta = 1$ give the following conditions under which the solution of (11) is sought.

TABLE 1

VALUE OF CRITICAL TAYLOR NUMBER T_c FOR DIFFERENT VALUES OF λ AND μ

λ	μ	$R=0,\ a=3\cdot12$ $\sigma=0$	$R=5\cdot0,\ a=3\cdot10$ $\sigma=-12\cdot6$	$R=20\cdot0,\ a=3\cdot40$ $\sigma=-55\cdot7$
0·0	1·0	1,701	1,740	2,310
	0·75	1,944	1,988	2,623
	0·50	2,258	2,305	2,951
	0·25	2,666	2,704	3,097
1·0	1·0	1,193	1,223	1,625
	0·75	1,367	1,397	1,844
	0·50	1,586	1,620	2,072
	0·25	1,874	1,903	2,182
2·0	1·0	928	970	1,287
	0·75	1,080	1,106	1,460
	0·50	1,257	1,284	1,643
	0·25	1,490	1,507	1,732
3·0	1·0	804	833	1,093
	0·75	921	940	1,236
	0·50	1,067	1,088	1,397
	0·25	1,262	1,281	1,469
—1·0	1·0	3,266	3,315	4,399
	0·75	3,701	3,784	4,986
	0·50	4,298	4,385	5,611
	0·25	5,065	5,141	5,895
—1·10	1·0	3,411	3,665	5,120
	0·75	4,091	4,184	5,512
	0·50	4,750	4,764	6,203
	0·25	5,601	5,684	6,521
—1·20	1·0	4,011	4,103	5,437
	0·75	4,583	4,684	6,173
	0·50	5,318	5,428	6,943
	0·25	6,270	6,364	7,298

$$\left.\begin{aligned} u &= 0 \\ Du &= 0 \\ D^4u - (2a^2 + iG)\,D^2u &= 0 \end{aligned}\right\} \text{at } \zeta = 0 \ \&\ \zeta = 1 \qquad (12)$$

Solution

We assume three solutions u_1, u_2 & u_3 of (11) for the prescribed values of μ, λ, σ, R and a, which satisfy the conditions (12) and in addition we assume

$$\left.\begin{aligned} D^2u_1 = 1\,, &\quad D^3u_1 = D^5u_1 = 0 \\ D^3u_2 = 1\,, &\quad D^2u_2 = D^5u_2 = 0 \\ D^5u_3 = 1\,, &\quad D^2u_3 = D^3u_3 = 0 \end{aligned}\right\} \text{at } \zeta = 0 \qquad (13)$$

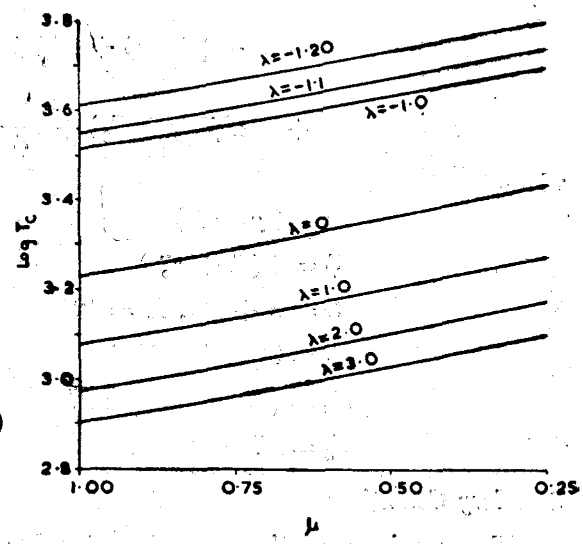

Fig. 1—The variation of the log of the critical Taylor number T_c with μ for $R=0$, $\sigma=0$ and $a=3.12$. The values of λ are shown on the curves.

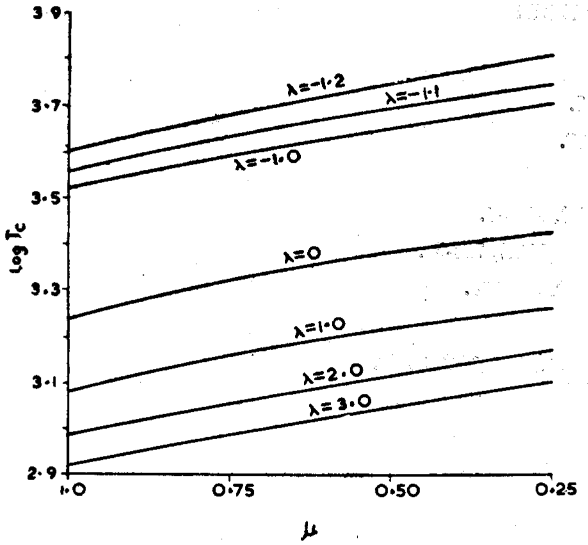

Fig. 2—The variation of the log of the critical Taylor number Tc with μ for R=5.0, $\sigma = -12.6$ and $a = 3.1$. The values of λ are shown on the curves.

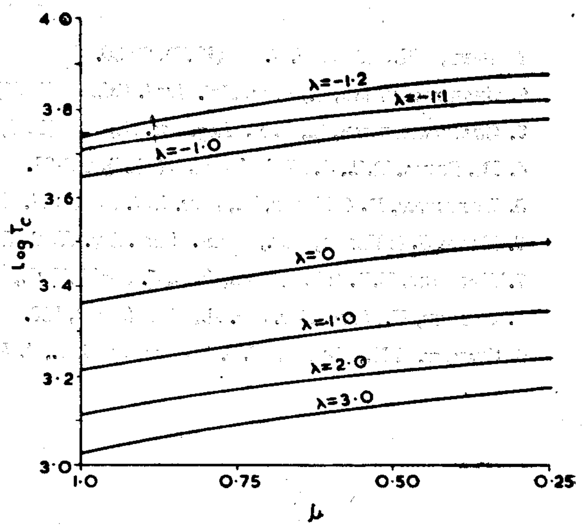

Fig. 3—The variation of the log of the critical Taylor number Tc with μ for $R = 20.0$, $\sigma = -55.70$ and $a = 3.40$. The values of λ are shown on the curves.

Now combining (12) and (13) we get six boundary conditions at $\zeta = 0$ for each of the solutions u_1, u_2 & u_3. The solution of (11), yielding the minimum value of T depicting the onset of instability, is found by a numerical technique based on Runge-Kutta method. The details of this numerical technique are given by Sparrow et. al.[9]

RESULTS

By assuming $\mu \geqslant 0$, we have computed the critical values T_c of Taylor number for $\lambda = 0$, 1, 2, 3, −1, −1·1 & −1.2 for Reynolds number $R = 0, 5$ & 20 and for different positive values of μ, the velocity ratio of the cylinders. The results have been presented in tabulated and graphical forms.

For the case of no radial velocity ($\lambda = 0$) and for $\mu = 1$ the results are in excellent agreement with those of Chandrasekhar[2]. The case $\mu = 1$ corresponds to the averaging of the angular velocity of the fluid in the annulus. In case of suction of fluid from the outer cylinder ($\lambda > 0$), the values of the critical Taylor numbers decrease for all values of R. Thus suction of fluid from the outer cylinder tends to damp out the disturbances with and without the presence of axial flow. Similarly injection of fluid ($\lambda < 0$) produces an opposite effect. The negative value of λ has to be less than 2 in magnitude, as had been observed by Bahl[1]. All the calculations have been carried out by fixing d/R_2 as 0.1.

It may be pointed out that by applying the same numerical technique, this problem can easily be solved even by taking the axial flow to be parabolic i.e. without averaging it. However, it has been observed by D. Prima[4] that the error introduced by the averaging of the axial flow is negligible for small Reynolds numbers, say up to 30.

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