LARGE DEFLECTION OF A TRIANGULAR ORTHOTROPIC PLATE ON ELASTIC FOUNDATION

S. Datta

Jalpaiguri Government Engineering College, Jalpaiguri

(*Received* 30 *July* 1974)

Large deflection of an equilateral triangular orthotropic plate, resting on elastic foundation has been solved for a uniform load throughout the plate. General expressions for deflection and bending moment at a particular point have been obtained and the limiting values of the theoretical results have been verified with the known results for small deflection and without any elastic foundation of the corresponding isotropic plate. Theoretical results have also been presented in the form of graphs.

Triangular reinforced concrete slabs are sometimes used as bottom slabs of bunkers. Thus the design of this type of structure is of practical interest for Defence. These slabs may rest freely on soil or sand and generally are subjected to a uniform load. If the thickness of the slab is small compared to the other dimensions, then it may be regarded as a thin orthotropic plate resting on elastic foundation and subjected to a uniform load. Within the elastic limit, the deflection of such plates may be large, i.e., the deflection is on the order of the thickness of the plate. When a plate undergoes large deflection, three differential equations for displacement and deflection may be written, but it is usually difficult to obtain solutions of these equations because of their non-linear character.

Various problems of large deflections of thin plates not resting on elastic foundation have been examined by Way[1], Levy[2] and many other authors. But the methods used by them involve and require considerable computation. Berger[3] suggested that the strain energy due to the second strain invariant of the middle surface strains may be neglected in analysing large deflection of plates having axis symmetric deformation. Berger's method reduces computation and although no complete explanation of this method is offered in, Berger has shown that the deflections and stresses obtained for circular plates under uniform load are in good agreement with those found in practical analysis. Since then numerous problems have been solved with remarkable case and satisfactory approximation by using this method. Iwinski and Nowinski[4] generalised the procedure of Berger to orthotropic plates and found out the deflections of circular and rectangular pl?tes under uniform load and various boundary conditions. By using this approximate method Banerjee[5] obtained deflections of a circular orthotropic plate under a concentrated load at the centre.

Berger's technique of neglecting the second strain invariant in the middle plane has been applied by Sinha[6] to determine large deflection of circular and rectangular plates under uniform load and resting on elastic foundation.

In this paper large deflection of an equilateral triangular orthotropic plate, such as reinforced concrete, resting on elastic foundation has been solved for a uniform load throughout the plate. Foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

NOTATIONS

a = one-half of the length of each side of the plate

e_1 = first invariant of middle surface strains

= $\epsilon_x + E_y$ in rectangular coordinates

h = plate thickness

K = foundation reaction per unit area per unit deflection

K_F = non-dimensional foundation modulus = $\frac{K}{D} a^4$

q = uniform lateral load

u, v = displacement along x and y direction respectively

V, V_1 = strain energy

w = deflection in z-direction

ϵ = direct strain

γ = shear strain

FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{1}$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2}$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \tag{3}$$

Neglecting the second middle surface strain invariant, the strain energy due to bending and stretching of the middle surface of the plate of thickness, h, can be written as

$$V_1 = \frac{1}{2}\iint\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4\,D_{xy}\left(\frac{\partial^2 w}{\partial x.\,\partial y}\right)^2 + \right.$$
$$\left. + D_x\frac{12}{h^2}e_1^2\right]dxdy \tag{4}$$

in which

$$D_x = \frac{E'_x h^3}{12},\ D_y = \frac{E'_y h^3}{12},\ D_1 = \frac{E''h^3}{12},\ D_{xy} = \frac{G\,h^3}{12} \tag{5}$$

$$e_1 = \frac{\partial u}{\partial x} + K_1\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K_1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{6}$$

$$K_1^2 = \frac{D_y}{D_x} \tag{7}$$

and E'_x, E'_y, E'', and G are constants to characterise the elastic properties of the material.

By adding the potential energy of the uniform normal load, 'q' and of the foundation reaction, K to the energy expression (4), the modified energy expression is obtained as follows:

$$V = \frac{1}{2}\iint\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{xy}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + \right.$$
$$\left. + D_x\frac{12}{h^2}e^2_1\right]dxdy - \iint qw\ dxdy + \frac{1}{2}\iint K\,w^2\,dxdy \tag{8}$$

According to the principle of minimum potential energy, the displacements satisfying the equilibrium conditions make the potential energy V minimum. In order for the integral of equation (8) to be an extremum, its integrand F, must satisfy the following Euler's variational principle:

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) = 0 \tag{9}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = 0 \tag{10}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \tag{11}$$

Applying (10) and (11) to (8) respectively, we get

$$\frac{\partial}{\partial x}(e_1) = 0 \tag{12}$$

$$\frac{\partial}{\partial y}(e_1) = 0 \tag{13}$$

Thus

$$e_1 = C \tag{14}$$

a normalised constant of integration to be determined. Applying (9) to (8) and considering (14), we get

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + \frac{2(D_1 + 2D_{xy})}{D_x} \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) + \frac{K}{D_x} = \frac{q}{D_x} \tag{15}$$

Introducing the notation

$$H = D_1 + 2\,D_{xy}$$

Equation (15) can be written as

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + 2\,\frac{H}{D_x} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12\,C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) + \frac{K}{D_x} = \frac{q}{D_x} \tag{16}$$

For a slab with two-way reinforcement in the directions x and y, H can be taken as[7]

$$H = (D_x\, D_y)^{\frac{1}{2}}$$

Introducing now

$$x_1 = x$$

$$y_1 = y\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} \tag{17}$$

Equation (16) is reduced to the form

$$(\nabla^2 - \alpha^2)\,\nabla^2\, w + \frac{K}{D_x}\, w = \frac{q}{D_x} \tag{18}$$

in which $$\alpha^2 = \frac{12\,C}{h^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

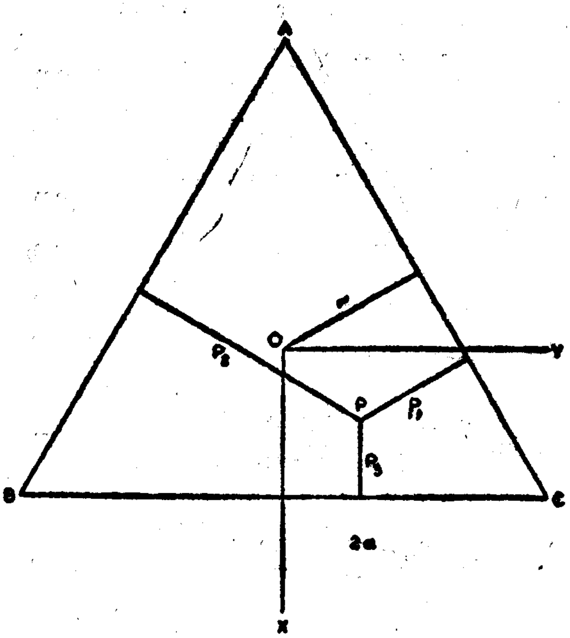

Fig. 1– Equilateral triangular orthotropic plate.

SOLUTION OF PROBLEM

Let the plate be in the form of an equilateral triangle, ABC (Fig. 1) having each side of length $2a$. Let the centroid O be the origin, X-axis and Y-axis perpendicular and parallel to the base BC respectively. If x_1, y_1 be the cartesian coordinates of any point, p, within the triangle p_1 p_2 p_3 be the three perpendiculars from P on CA, AB and BC respectively, and r be the radius of the inscribed circle, then

$$P_1 = r + \frac{x_1}{2} - \frac{y_1\sqrt{3}}{2},$$

$$P_2 = r + \frac{x_1}{2} + \frac{y_1\sqrt{3}}{2},$$

$$P_3 = r - x_1 \ ,$$
$$P_1 + P_2 + P_3 = 3r = \sqrt{3}a = K_2 = \text{constant},$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$
$$= \frac{\partial^2}{\partial P_1^2} + \frac{\partial^2}{\partial P_2^2} + \frac{\partial^2}{\partial P_3^2} - \frac{\partial^2}{\partial P_1 \partial P_2} - \frac{\partial^2}{\partial P_2 \partial P_3} - \frac{\partial^2}{\partial P_3 \partial P_1}$$

Using the trilinear Coordinates[8] p_1, p_2, p_3 the deflection w can be taken in the form

$$w = \sum_{n=1}^{\infty} A_n \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{19}$$

where $A_n =$ a constant.

The above form of w satisfies the following boundary conditions of simply supported edges :

$$w = 0 \text{ at } P_1 = 0, \ P_2 = 0, \ P_3 = 0$$

Expanding the transverse uniform load q, into Fourier Sine series

$$q = \sum_{n=1}^{\infty} \frac{2q}{n\pi} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{20}$$

and substituting (19) and (20) into (18), we get

$$A_n = \sum_{n=1}^{\infty} \frac{2q}{n\pi D_x} \cdot \frac{1}{\left[\left(\frac{2n\pi}{K_2} \right) + \alpha^2 \left(\frac{2n\pi}{K_2} \right) + \frac{K}{D_x} \right]} \tag{21}$$

To determine α, Equation (6) is transformed into x_1, y_1 coordinates in the following form

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x_1} + (K_1)^{\frac{1}{2}} \frac{\partial v}{\partial y_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y_1} \right) \tag{22}$$

The boundary conditions on u and v are

$$u = 0 \text{ at } P_3 = 0 \tag{23}$$

$$\sqrt{3}\, v + u = 0 \text{ at } P_2 = 0 \tag{24}$$

$$\sqrt{3}\, v - u = 0 \text{ at } P_1 = 0 \tag{25}$$

The following forms of u and v satisfy the above boundary conditions.

$$u = \sum_{m=1}^{\infty} \sqrt{3}\, B_m \left[\sin \frac{2_m \pi (P_2 + P_3)}{K_2} + \sin \frac{2_m \pi (P_1 + P_3)}{K_2} \right] \tag{26}$$

$$v = \sum_{m=1}^{\infty} \frac{1}{\sqrt{K_1}} B_m \left[\sin \frac{2_m \pi (P_1 + P_3)}{K_2} - \sin \frac{2_m \pi (P_2 + P_3)}{K_2} \right] \tag{27}$$

in which B_m is a constant.

Substituting the expressions for u, v and w into (22) and integrating over the whole area of the plate, the following equation determining α is obtained:

$$\frac{\alpha^2 h^2}{12} = \sum_{n=1}^{\infty} \frac{3 A_n^2 n^2 \pi^2}{K_2^2} \tag{28}$$

Thus w is completely determined in the following form in x, y coordinates

$$w = A_n \left[2 \sin 2n\pi \left(\frac{1}{3} + \frac{x}{2\sqrt{3}a} \right) \cos \frac{2n\pi y}{2\sqrt{K_1}\, a} + \sin 2n\pi \left(\frac{1}{3} - \frac{x}{\sqrt{3}a} \right) \right] \tag{29}$$

If $D_x = D_y = D$, $\alpha \to 0$, and $K = 0$, (19) and (21) give the deflection equation for an isotropic plate not resting on the elastic foundation in the following form :

$$w = \sum_{n=1}^{\infty} \frac{q K_2^4}{8 n^5 \pi^5 D} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \tag{30}$$

The corresponding equation as obtained by S. Woinowsky-Krieger for a plate having each side of length $\frac{2a}{\sqrt{3}}$ is

$$w = \frac{q}{64aD} \left[x^3 - 3y^2 x - a(x^2 + y^2) + \frac{4}{27} a^3 \right] \left(\frac{4}{9} a^2 - x^2 - y^2 \right) \tag{31}$$

At the origin ($P_1 = P_2 = P_3$), w is given by (30) as

$$w = \frac{27 q a^4}{8\pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin \frac{2n\pi}{3} = \cdot 009 \frac{qa^4}{D} \tag{32}$$

which is numerically equal to that obtained from (31) for the plate having each side of length $2a$ as

$$(w)_{x=y=0} = \frac{qa^4}{108D} = \cdot 009 \frac{qa^4}{D}$$

NUMERICAL CALCULATION

To calculate deflection at any point within the plate, we have to start from (28) with an assumed value of ($\alpha\, a$) leading to the corresponding value of the load function $\frac{qa^4}{D_x h}$. Once this relationship is obtained, the corresponding deflection can be obtained from (19) with the help of (21).

At the origin maximum deflection is obtained and is given by

$$\frac{w_{max}}{h} = \frac{6}{\pi} \left(\frac{qa^4}{D_x h} \right) \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{3}}{n \left[\frac{16 \pi^4 n^4}{9} + \frac{4 \pi^2 n^2 \alpha^2 a^2}{3} + K_F \right]} \tag{33}$$

in which the non-dimensional foundation modulus

$$K = \frac{K a^4}{D_x} \tag{34}$$

For $K_F = 0$ and $K_F = 100$ graphs are plotted in Fig. 2 showing the deflection $\frac{w}{h}$ at the centroid of the plate against the loads. Fig 2. also contains a graph plotted according to the linear theory.

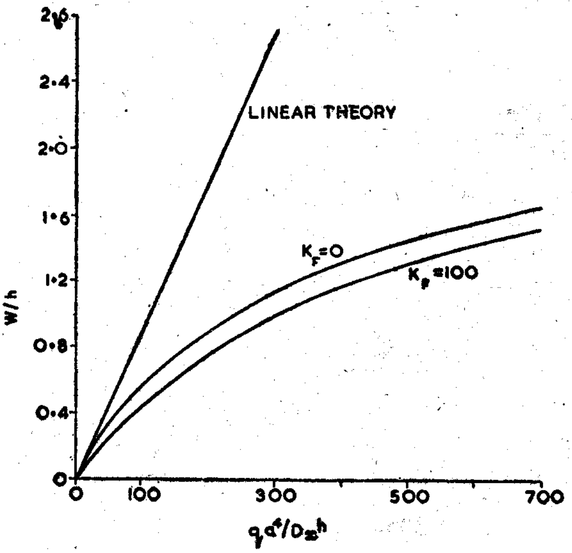

Fig. 2– Deflection curve.

CONCLUSION

From Fig. 2 it is clear that design calculations should be made according to the non-linear theory, because deflections calculated according to small deflection theory will be far from the actual values for higher values of load function. The effect of the foundation is to reduce the deflection for a given value of load function.

Because the deflection, w, has been determined, bending moments and stresses can be computed easily. The bending moments M_x and M_y at the centroid of the plate are obtained as

$$M_x = 4\,(1+\nu_c)\,qa^2 \sum_{n=1}^{\infty} \frac{n \sin \dfrac{2n\pi}{3}}{\left[\dfrac{16\,\pi^4 n^4}{9} + \dfrac{4\,\pi^2 n^2 \alpha^2 a^2}{3} + K\right]} \tag{35}$$

$$M_y = K_1\, M_x \tag{36}$$

ν_c is the Poisson's ratio for concrete.

For isotropic plate without elastic foundation and undergoing small deflection $\nu_c = \nu$, $K_1 = 1$, $K_F = 0$, $\alpha \to 0$ and for a plate having each side of length $\dfrac{2a}{\sqrt{3}}$, (35) and (36) lead to

$$M_x = M_y = (1+\nu)\,\frac{q\,a^2}{54} \tag{37}$$

which is the same result obtained by Timoshenko[7].

ACKNOWLEDGEMENT

The author wishes to thank Dr. B. Banerjee, Jalpaiguri Government Engineering College for his help and guidance in preparation of this paper.

REFERENCES

1. WAY, S., *Transactions ASME*, **56** (1934), 627.
2. LEVY, S., "Bending of Rectangular Plates with Large Deflections", Technical Note No. 846, NACA, Washington, D.C., May 1942.
3. BERGER, H.M., *Journal of Applied Mechanics, ASME*, **22** (1955), 465-472.
4. IWINSKI, T. & NOWINSKI, J., *Archiwum Mechaniki Stosowanej*, **9** (1957), 953-603.
5. BANERJEE, B., *Bulletin De L' academic, Polonaise Des Sciences*, **15** (1967), 699-704.
6. SINHA, S.N., *Journal of the Engineering Mechanics Division, Proceedings of the ASCE*, 89, No. EM1, 1—24.
7. TIMOSHENKO, S. & WOINOWSKY-KRIEGER, S., "Theory of Plates and Shells' (McGraw-Hill Book Co., Inc, New York) 2nd Edition, 1959, p. 366, 313 & 314.
8. SEN, B., *Bulletin of Calcutta Mathematical Society*, **60** (1968), 25-30.