APPLICATION OF GENERALIZED FUNCTION IN THE PRODUCTION OF HEAT IN A CYLINDER

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The present paper deals with the application of H-function of two variables in solving the fundamental differential equation of diffusion of heat in a cylinder.

We have employed H-function of two variables defined by Mittal and Gupta[1] to solve the fundamental differential equation of the diffusion of heat in a cylinder of radius y when there are sources of heat within it which lead to an axailly symmetrical temperature distribution. The fundamental differential equation given by Sneddon[2] is

$$\frac{\partial \phi}{\partial t} = \frac{K}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \phi}{\partial x} \right) + \theta (x, t) \tag{1}$$

We assume that the rate of generation of heat is independent of temperature and the cylinder is infinitely long so that the variation of Z may be neglected. We shall, in addition, suppose that the surface $x = y$ is maintained at zero temperature and initial distribution temperature is also zero. We further suppose that

$$\theta (x, t) = \frac{k}{K} f (x) \, g (t) \tag{2}$$

where k is the diffusivity and K the conductivity of the material. It will be observed that the single function $f(x)$ can represent both sources and sinks embedded in the system. Whenever the product $f(x)$ $g(t)$ gives a negative value, it should be treated as sink. We shall characterise the heat sources by the behaviour of the function $g(t)$.

The following formula due to Ram[7] is required in this paper.

$$\int_0^y x^{\rho-1} (y-x)^{\mu-1} H_{p,\, q+1}^{m+1,\, n} \left[a x^{\sigma} (y-x)^{\sigma_1} \middle| \begin{matrix} \{(a'_p, \alpha'_p)\} \\ (b_o, \beta_o), \{(b'_q, \beta'_q)\} \end{matrix} \right] \times$$

$$\times H \left[\begin{matrix} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix} \end{matrix} \middle| \begin{matrix} \{(a_{p1}, \alpha_{p1}, A_{p1})\} \\ \{(b_{q1}, \beta_{q1}, B_{q1})\} \\ \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \\ \{(e_{p3}, E_{p3})\} \\ \{(f_{q3}, F_{q2})\} \end{matrix} \middle| \, b x^h (y-x)^{h_1}, \; c x^k (y-x)^{k_1} \right] dx$$

$$= \frac{y^{\rho+\mu-1}}{\beta_o} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\prod_{j=1}^{m} \Gamma(b'_j - \beta'_j \rho_r) \prod_{j=1}^{n} \Gamma(1 - a'_j + \alpha'_j \rho_r)}{\prod_{j=m+1}^{q} \Gamma(1 - b'_j + \beta'_j \rho_r) \prod_{n+1}^{p} \Gamma(a'_j - \alpha'_j \rho_r)} a^{\rho_r} y^{(\sigma+\sigma_1)\rho_r} \times$$

$$\times H \left[\begin{matrix} \begin{pmatrix} 0, & n_1+2 \\ p_1+2, & q_1+1 \end{pmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ q_2, & q_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} (1-\rho-\sigma\rho_r, h, k), (1-\mu-\sigma_1\rho_r, h_1, k_1), \{(a_{p_1}, \alpha_{p_1}, A_{p_1})\} \\ (1-\rho-\mu-(\sigma+\sigma_1)\rho_r, h+h_1, k+k_1), \{(b_{q_1}, \beta_{q}, B_{q_1})\} \\ \{(c_{p_2}, \gamma_{p_2})\} \\ \{(d_{q_2}, \delta_{q_2})\} \\ \{(e_{p_3}, E_{p_3})\} \\ \{(f_{q_3}, F_{q_3})\} \end{matrix} \right| \begin{matrix} by^{h+h_1} \\ cy^{k+k_1} \end{matrix} \right] \tag{3}$$

where

$$\rho_r = \frac{b_o+r}{\beta_o}, \quad \beta_o > 0$$

$$\beta < R\,(b_o/\beta_o) < \delta, |\arg a| < \tfrac{1}{2}\lambda\pi, \lambda > 0, A > 0, \sigma, \sigma_1, h, h_1, k, k_1 \geqslant 0,$$

$$R\left(\rho + \sigma\,\frac{b_o}{\beta_o} + h\alpha' + k\beta'\right) > 0, \; R\left(\mu + \sigma_1\,\frac{b_o}{\beta_o} + h_1\alpha' + k_1\beta'\right) > 0,$$

where

$$\delta = \min R\,(b'_j/\beta'_j),\; j = 1, 2, \ldots, m \tag{4}$$

$$\beta = \max R\left(\frac{a'_i - 1}{\alpha'_i}\right),\; i = 1, 2, \ldots, n \tag{5}$$

$$\lambda = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j \tag{6}$$

$$A = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \tag{7}$$

$$\alpha' = \min R\,(d_j/\delta_j),\; j = 1, \ldots, m_2 \tag{8}$$

$$\beta' = \min R\,(f_j/F_j),\; j = 1, 2, \ldots, m_3, \text{ and} \tag{9}$$

$$\sum_1^{p_1} \alpha_j + \sum_1^{p_2} \gamma_j < \sum_1^{q_1} \beta_j + \sum_1^{q_2} \delta_j \tag{10}$$

$$\sum_1^{p_1} A_j + \sum_1^{p_3} E_j < \sum_1^{q_1} B_j + \sum_1^{q_3} F_j \tag{11}$$

$$u = \sum_1^{n_1} \alpha_j - \sum_{n_1+1}^{p_1} \alpha_j - \sum_1^{q_1} \beta_j + \sum_1^{m_2} \delta_j - \sum_{m_2+1}^{q_2} \delta_j + \sum_1^{n_2} \gamma_j - \sum_{n_2+1}^{p_2} \gamma_j > 0 \tag{12}$$

$$|\arg b| < \tfrac{1}{2} u \pi \tag{13}$$

$$v = \sum_1^{n_1} A_j - \sum_{n_1+1}^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_3} F_j - \sum_{m_3+1}^{q_3} F_j + \sum_1^{n_3} E_j - \sum_{n_3+1}^{p_3} E_j > 0 \tag{14}$$

$$|\arg c| < \tfrac{1}{2} v \pi. \tag{15}$$

The proof of the formula (3) follows from the series expansion of

$$H_{p,\, q+1}^{m+1,\, n}\left[ax^{\sigma}(y-x)^{\sigma_1} \left| \begin{matrix} \{(a'_p, \alpha'_p)\} \\ (b_o, \beta_o), \{(b'_q, \beta'_q)\} \end{matrix} \right. \right]$$

given by Mukherjee and Prasad[8] and the definition of H-function of two variables into contour integral form.

Particular Case :

(*i*) Putting $p_1 = q_1 = 0$, we get

$$\int_0^y x^{\rho-1}(y-x)^{\mu-1} H^{m+1,\,n}_{p,\,q+1}\left[ax^{\sigma}(y-x)^{\sigma_1}\left|\begin{matrix}\{(a'_p,\alpha'_p)\}\\(b_o,\beta_o),\{(b'_q,\beta'_q)\}\end{matrix}\right.\right]\times$$

$$\times H^{m_2,n_2}_{p_2,q_2}\left[bx^h(y-x)^{h_1}\left|\begin{matrix}\{(c_{p_2},\gamma_{p_2})\}\\\{(d_{q_2},\delta_{q_2})\}\end{matrix}\right.\right] H^{m_3,n_3}_{p_3,q_3}\left[cx^k(y-x)^{k_1}\left|\begin{matrix}\{(e_{p_3},E_{p_3})\}\\\{(f_{q_3},F_{q_3})\}\end{matrix}\right.\right]dx$$

$$=\frac{y^{\rho+\mu-1}}{\beta_o}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\prod_{j=1}^{m}\Gamma(b'_j-\beta'_j\rho_r)\prod_{j=1}^{n}\Gamma(1-a'_j+\alpha'_j\rho_r)}{\prod_{j=m+1}^{q}\Gamma(1-b'_j+\beta'_j\rho_r)\prod_{j=n+1}^{p}\Gamma(a'_j-\alpha'_j\rho_r)}a^{\rho_r}y^{(\sigma+\sigma_1)\rho_r}\times$$

$$\times H\left[\begin{matrix}\begin{pmatrix}0, & 1\\2, & 1\end{pmatrix}\\\begin{pmatrix}m_2, & n_2\\p_2, & q_2\end{pmatrix}\\\begin{pmatrix}m_3, & n_3\\p_3, & q_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(1-\rho-\sigma\rho_r,h,k),(1-\mu-\sigma_1\rho_r,h_1,k_1)\\(1-\rho-\mu-(\sigma+\sigma_1)\rho_r,h+h_1,k+k_1)\\\{(c_{p_2},\gamma_{p_2})\}\\\{(d_{q_2},\delta_{q_2})\}\\\{(e_{p_3},E_{p_3})\}\\\{(f_{q_3},F_{q_3})\}\end{matrix}\right|by^{h+h_1},cy^{k+k_1}\right] \qquad (16)$$

provided that the conditions given above with $p_1 = q_1 = 0$ are satisfied.

(*ii*) Putting $m = n = p = 0$, $q = 1$, $b_1 = b_o = 0$, $\beta_o = 1 = \beta'_1$, $\sigma_1 = 0$, $a = w_i^2/4$, $\sigma = 2$ in (3) and using Erdelyi[9], we have

$$\int_o^y x^{\rho-1}(y-x)^{\mu-1} j_o(w_i x)\, H\{bx^h(y-x)^{h_1}, cx^k(y-x)^{k_1}\}\,dx =$$

$$= y^{\rho+\mu-1}\sum_{r=0}^{\infty}\frac{(-r)^r}{r!\,\Gamma(1+r)}\left(\tfrac{1}{4}w_i^2\right)^r y^{2r}\times$$

$$\times H\left[\begin{matrix}\begin{pmatrix}0, & n_1+2\\p_1+2, & q_1+1\end{pmatrix}\\\begin{pmatrix}m_2, & n_2\\p_2, & q_2\end{pmatrix}\\\begin{pmatrix}m_3, & n_3\\p_3, & q_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(1-\rho-2r,h,k),(1-\mu,h_1,k_1),\{(a_{p1},\alpha_{q1},A_{p1})\}\\(1-\rho-h-2r,h+h_1,k+k_1),\{(b_{q1},\beta_{q_1},B_{q1})\}\\\{(c_{p_2},\gamma_{p2})\{\\\{d_{q2},\delta_{q2})\}\\(\{e_{p3},E_{p3})\}\\\{(f_{q_3},F_{q_3})\}\end{matrix}\right|by^{h+h_1},cy^{k+k_1}\right] \qquad (17)$$

provided that $h, h_1, k, k_1 \geqslant 0$, $R(\rho + h\alpha' + k\beta') > 0$,

$R(\mu + h_1\alpha' + k_1\beta') > 0$, where α', β' are given by (8) and (9) and the condition given by (10) to (15), are satisfied where $H\{bx^h(y-x)^{h}{}_1, cx^k(y-x)^{k_1}\}$ stands for H-function of two variable involved in (3).

(*ii*) Putting $m = n = p = q = 0$, $b = 0$, $\beta_o = 1$ in (3) and taking $a \to 0$ in the expanded form of right hand side of (3), we get

$$\int_0^y x^{\rho-1}(y-x)^{\mu-1} H\left[\begin{matrix}\begin{pmatrix}0, n_1\\ p_1, q_1\end{pmatrix}\\ \begin{pmatrix}m_2, n_2\\ p_2, q_2\end{pmatrix}\\ \begin{pmatrix}m_3, n_3\\ p_3, q_3\end{pmatrix}\end{matrix}\middle|\begin{matrix}\{(a_{p1}, \alpha_{p1}, A_{p1})\}\\ \{(b_{q1}, \beta_{q1}, B_{q1})\}\\ \{(c_{p2}, \gamma_{p2})\}\\ \{(d_{q2}, \delta_{q2})\}\\ \{(e_3, E_{p3})\}\\ \{(f_{q3}, F_{q3})\}\end{matrix}\middle| bx^h (y-x)^{h_1}, cx^k (y-x)^{k_1}\right] dx$$

$$= y^{\rho+\mu-1} H\left[\begin{matrix}\begin{pmatrix}0, n_1+2\\ p_1+2, q_1+1\end{pmatrix}\\ \begin{pmatrix}m_2, n_2\\ p_2, q_2\end{pmatrix}\\ \begin{pmatrix}m_3, n_3\\ p_3, q_3\end{pmatrix}\end{matrix}\middle|\begin{matrix}(1-\rho, h, k), (1-\mu, h_1 k_1), \{(a_{p1}, \alpha_{p1}, A_{p1})\}\\ (1-\rho-\mu, h+h_1, k+k_1), \{(b_{q1}, \beta_{q1}, B_{q1})\}\\ \{(c_{p2}, \gamma_{p2})\}\\ \{(d_{q2}, \delta_{q2})\}\\ \{(e_{p3}, E_{p3})\}\\ \{(f_{q3}, F_{q3})\}\end{matrix}\middle| by^{h+h_1}, cy^{k+k_1}\right] \quad (18)$$

where $h, h_1, k, k_1 > 0, R(\rho + h\alpha' + k\beta') > 0, R(\mu + h_1\alpha' + k_1\beta') > 0$, and conditions from (4) to (15) are satisfied.

Finite Hankel Transform:

Let the finite Hankel transform[2] of $f(x)$ be

$$\bar{f}_j(w_i) = \int_0^y x f(x) J_o(x w_i)\, dx \quad (19)$$

where w_i is the root of the transcendental equation

$$J_o(y w_i) = 0 \quad (20)$$

where $f(x) = x^{\rho-2}(y-x)^{\mu-1} H\{bx^h (y-x)^{h_1}, cx^k (y-x)^{k_1}\}$ in (19) and using the result (17), we get

$$\bar{f}_j(w_i) = y^{\rho+\mu-1} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{w_i y}{2}\right)^{2r} \times$$

$$\times H\left[\begin{matrix}\begin{pmatrix}0, n_1+2\\ p_1+2, q_1+1\end{pmatrix}\\ \begin{pmatrix}m_2, n_2\\ p_2, q_2\end{pmatrix}\\ \begin{pmatrix}m_3, n_3\\ p_3, q_3\end{pmatrix}\end{matrix}\middle|\begin{matrix}(1-\rho-2r, h, k), (1-\mu, h_1, k_1), \{(a_{p1}, \alpha_{p1}, A_{p1})\}\\ (1-\rho-h-2r, h+h_1, k+k_1), \{(b_{q1}, \beta_{q1}, B_{q1})\}\\ \{(c_{p2}, \gamma_{p2})\}\\ \{(d_{q2}, \delta_{q2})\}\\ \{(e_{p3}, E_{p3})\}\\ \{(f_{q3}, F_{q3})\}\end{matrix}\middle| by^{h+h_1}, cy^{k+k_1}\right] \quad (21)$$

where conditions given in (17) are satisfied.

By virtue of the inversion theorem[2]

$$f(x) = x^{\rho-2}(y-x)^{\mu-1} H\{bx^h (y-x)^{h_1}, cx^k (y-x)^{k_1}\}$$

$$= 2y^{\rho+\mu-3} \sum_i \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\tfrac{1}{2} w_i y\right)^{2r} \frac{J_o(x w_i)}{[J_1(y w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0, n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1), \{(a_{p1}, \alpha_{p1}, A_{p1})\} \\ (1-\rho-h-2r, h+h_1, k+k_1), \{(b_{q1}, \beta_{q1}, B_{q1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \end{array} & by^{h+h_1}, cy^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p3}, E_{p3})\} \\ \{(f_{q3}, F_{q3})\} \end{array} & \end{array} \right] \quad (22)$$

where the sum is taken over all positive roots of (20). The result (22) will be proved useful in the verification of the solutions.

SOLUTION OF THE PROBLEM

We apply finite Hankel transform (21) to obtain the solution of (1). Its solution obtained as in Sneddon[9],

when

$$\theta(x,t) = \frac{k}{K} f(x)\, g(t)$$

where $f(x)$ is a function of x alone and $g(t)$ is a function of t alone, is

$$\Phi(x,t) = \frac{2k}{K} y^{\rho+\mu-3} \sum_i \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\tfrac{1}{2} w_i y\right)^{2r} \frac{J_0(x w_i)}{[J_1(y w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0, n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1), \{(a_{p1}, \alpha_{p1}, A_{p1})\} \\ (1-\rho-2r, h+h_1, k+k_1), \{(b_{q1}, \beta_{q1}, B_{q1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \end{array} & by^{h+h_1}, cy^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p3}, E_{p3})\} \\ \{f_{q3}, F_{q3})\} \end{array} & \end{array} \right] \psi(w_i,t) \quad (23)$$

where

$$\psi(w,t) = \int_0^t g(T)\, e^{-kwi^2 (t-T)}\, dT \quad (24)$$

provided that the conditons given in (17) are satisfied.

VERIFICATION OF THE SOLUTION

From (23), we have

$$\frac{k}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \phi}{\partial x} \right) = -\frac{2k^2}{K} y^{\rho+\mu-3} \sum_i \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\tfrac{1}{2} w_i y\right)^{2r} w_i^2 \cdot \frac{J_0(w_i x)}{[J_1(y w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0, n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1), \{(a_{p1}, \alpha_{p1}, A_{p1})\} \\ (1-\rho-h-2r, h+h_1, k+k_1), \{(b_{q1}, \beta_{q1}, B_{q1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \end{array} & by^{h+h_1}, cy^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p3}, E_{p3})\} \\ \{(f_{q3}, F_{q3})\} \end{array} & \end{array} \right] \times$$

$$\times \int_0^t g(T)\, e^{-kw_i^2(t-T)}\, dT. \tag{25}$$

From (2) and (22), we get

$$\theta(x,t) = 2\,\frac{k}{K}\, y^{\rho+\mu-3} \sum_i \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\tfrac{1}{2} w_i\, y\right)^{2r} \frac{J_0(x\, w_i)}{[J_1(y\, w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0,\ n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1)\ \{(a_{p1}, \alpha_{p1}\, A_{p1})\} \\ (1-\rho-h-2r, h+h_1, k+k_1),\ \{(b_{q1}, \beta_{q1}, B_{q1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \end{array} & by^{h+h_1},\ cy^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p3}, E_{p3})\} \\ \{(f_{q3}, F_{q3})\} \end{array} & \end{array} \right] g(t) \tag{26}$$

From (23), we have

$$\frac{\partial\phi}{\partial t} = \frac{2k}{K}\, y^{\rho+\mu-3} \sum_i \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\tfrac{1}{2} w_i\, y\right)^{2r} \frac{J_0(x\, w_i)}{[J_1(y\, w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0,\ n_1+2 \\ p_1+2,\ q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1),\ \{(a_{p1}, \alpha_{p1}, A_{p1})\} \\ (1-\rho-h-2r, h+h_1, k+k_1),\ \{(b_{q1}, \beta_q, B_{q1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p2}, \gamma_{p2})\} \\ \{(d_{q2}, \delta_{q2})\} \end{array} & by^{h+h_1},\ cy^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3\ q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p3}, E_{p3})\} \\ \{(f_{q3}, F_{q3})\} \end{array} & \end{array} \right] \times$$

$$\times \left[g(t) - k\, w_i^2 \int_0^t g(t)\, e^{-kw_i^2(t-T)}\, dT \right]. \tag{27}$$

Substituting the above values in (1), the equation is staisfied. The boundary condition $\phi(y,t) = 0$ is, satisfied because $J_0(y\, w_i)$ which is present in every term of $\phi(y, t)$ is zero. The initial condition is satisfied because $\psi(w_i, 0) = 0$. We see that (23) converges uniformly when $t > 0$ and so the function $\phi(x, t)$ represented by it is continuous when $\theta \leqslant x \leqslant y$. The term by term differentiations are justified because (25) and (27) are uniformly convergent, when $t > 0$ and $0 \leqslant x \leqslant y$.

HEAT SOURCE

Heat source of general character: Let the function $g(T)$ be

$$g(T) = T^{\rho'-1}\,(t-T)^{\mu'-1} \times$$

$$\times H \left[\begin{array}{c|l|c} \begin{pmatrix} 0,\ n'_1 \\ p_1', q'_1 \end{pmatrix} & \begin{array}{l} \{(a'_{p'1}, \alpha'_{p'1}, A'_{p'1})\} \\ \{(b'_{q'1}, \beta'_{q'1}, B'_{q'1})\} \end{array} & \\ \begin{pmatrix} m'_2, n'_2 \\ p'_2, q'_2 \end{pmatrix} & \begin{array}{l} \{(c'_{p'2}, \alpha'_{p'2})\} \\ \{(d'_{q'2}, \delta'_{q'2})\} \end{array} & b'\, T^{h'}\,(t-T)^{h'_1},\ c'\, T^{k'}\,(t-T)^{k'_1} \\ \begin{pmatrix} m'_3, n'_3 \\ p'_3, q'_3 \end{pmatrix} & \begin{array}{l} \{(e'_{p'3}, \{(E'_{p'3})\} \\ \{(f'_{q'3}, F'_{q'3})\} \end{array} & \end{array} \right] \tag{28}$$

So

$$\psi(w_i, t) = \int_0^t g(T)\, e^{-kw_i^2(t-T)}\, dT$$

$$= \int_0^t T^{\rho'-1}(t-T)^{\mu'-1} e^{-kw_i^2(t-T)} H\left\{ b' T^{h'}(t-T)^{h'_1},\ c' T^{k'}(t-T)^{k'_1} \right\} dT$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (k u_i^2)^N \int_0^t T'^{\rho-1} (t-T)^{N+\mu''-1} H\{ b' T^{h'} (t-T)^{h'_1}, c' T^{k'} (t-T)^{k'_1}\} dT$$

$$= t^{\rho'+\mu'-1} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (k+w_i^2)^N \times$$

$$\times H \left[\begin{array}{c|l|l} \begin{pmatrix} 0, n'_1+2 \\ p'_1+2, q'_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho', h', k'), (1-\mu'-N, h'_1, k'_1), \{(a'_{p'_1}, \alpha'_{p'_1}, A'_{p'_1})\} \\ (1-\rho'-\mu'-N, h'+h'_1, k'+k'_1), \{(b'_{q'_1}, \beta'_{q'_1}, B'_{q'_1})\} \end{array} & \\ \begin{pmatrix} m'_2, n'_2 \\ p'_2, q'_2 \end{pmatrix} & \begin{array}{l} \{(c'_{p'_2}, \gamma'_{p'_2})\} \\ \{(d'_{q'_2}, \delta'_{q'_2})\} \end{array} & \begin{array}{l} b'\, t^{h'+h'_1}, \\ c'\, t^{k'+k'_1} \end{array} \\ \begin{pmatrix} m'_3, n'_3 \\ p'_3, q'_3 \end{pmatrix} & \begin{array}{l} \{(e'_{p'_3}, E'_{p'_3})\} \\ \{(f'_{q'_3}, F'_{q'_3})\} \end{array} & \end{array} \right] \tag{29}$$

provided that $h, h_1, k, k_1 > 0$, $R(\rho' + h'\alpha'' + k'\beta'') > 0$, $R(\mu' + h'_1\alpha'' + k'_1\beta'') > 0$ where α'', β' are given by equations (8) and (9) with d, δ, f, F, m_2 and m_3 replaced by $d', \delta', f', F', m'_2$ and m'_3 and equations (10) to (15) with all letters, replaced by their dashes are satisfied where $H\{b' T^{h'} (t-T)^{h'}, c' T^{k'} (t-T)^{k'_1}\}$ stands for H-function of two variables involved in (28). From (23) and (29), the solution is

$$\phi(x, t) = \frac{2k}{K} y^{\rho+\mu-3}\, t^{\rho'+\mu'-1} \sum_i \sum_{r=0}^{\infty} \sum_{N=0}^{\infty} \frac{(-1)^{r+N} (\tfrac{1}{2}y)^{2r}}{(r!)^2 N!} (kt)^N (w_i)^{2(r+n)} \frac{J_0(x w_i)}{[J_1(y w_i)]^2} \times$$

$$\times H \left[\begin{array}{c|l|l} \begin{pmatrix} 0, n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho-2r, h, k), (1-\mu, h_1, k_1), \{(a_{p_1}, \alpha_{p_1}, A_{p_1})\} \\ (1-\rho-h-2r, h+h_1, k+k_1), \{(b_{q_1}, \beta_{q_1}, B_{q_1})\} \end{array} & \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} & \begin{array}{l} \{(c_{p_2}, \gamma_{p_2})\} \\ \{(d_{q_2}, \delta_{q_2})\} \end{array} & b y^{h+h_1}, c y^{k+k_1} \\ \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} & \begin{array}{l} \{(e_{p_3}, E_{p_3})\} \\ \{(f_{q_3}, F_{q_3})\} \end{array} & \end{array} \right] \times$$

$$\times H \left[\begin{array}{c|l|l} \begin{pmatrix} 0, n'_1+2 \\ p'_1+2, q'_1+1 \end{pmatrix} & \begin{array}{l} (1-\rho', h', k'), (1-\mu'-N, h'_1, k'_1), \{(a'_{p'_1}, \alpha'_{p'_1}, A'_{p'_1})\} \\ (1-\rho'-\mu'-N, h'+h'_1, k'+k'_1), \{(b'_{q'_1}, \beta'_{q'_1}, B'_{q'_1})\} \end{array} & \\ \begin{pmatrix} m'_2, n'_2 \\ p'_2, q'_2 \end{pmatrix} & \begin{array}{l} \{(c'_{p'_2}, \gamma'_{p'_2})\} \\ \{(d'_{q'_2}, \delta'_{q'_2})\} \end{array} & b' t^{h'+h'_1}, c t^{k'+k'_1} \\ \begin{pmatrix} m'_3, n'_3 \\ p'_3, q'_3 \end{pmatrix} & \begin{array}{l} \{(e'_{p'_3}, E'_{p'_3})\} \\ \{(f'_{q'_3}, E'_{p'_3})\} \end{array} & \end{array} \right] \tag{30}$$

provided that the conditions given in (23) and (29) are satisfied. Obviously $\phi(x, o) = 0$.

Particular Cases :

(*i*) Putting $p_1 = p'_1 = q'_1 = q_1 = q'_1 = 0 = h = k_1 = h' = k'_1$, $m_2 = 1 = m'_2$, $n_2 = n'_2 = p_2 = p'_2 = 2 = q_2 = q'_2$, $\delta_1 = \delta'_1 = \delta_2 = \delta'_2 = 1$, $\gamma_1 = \gamma'_1 = 1 = \gamma_2 = \gamma'_2$, $m_3 = 1$, $n_3 = p_3 = u$, $q_3 = v + 1$, $E_1 = E_2 = \ldots = E_{P_3} = 1 = F_1 = F_2 = \ldots = F_{q_3}$, $f_1 = 0$, and replacing e_{P_3} and f_{q_3} by $1 - e_{p_3}$ and $1 - f_{q_3}$, in (30) we obtain the result recently obtained by Singh[3]. equation (18).

(*ii*) Putting $p_1 = p'_1 = q_1 = q'_1 = 0 = h = k_1 = h' = k'_1$, $m_2 = 1 = m'_2$, $n_2 = n'_2 = p_2 = p'_2 = 2 = q_2 = q'_2$, $\delta_1 = \delta'_1 = \delta_2 = \delta'_2 = 1 = \gamma_1 = \gamma'_1 = \gamma_2 = \gamma'_2$, $E_1 = E_2 = \ldots = E_{P_3} = 1 = F_1 = F_2 = \ldots = F_{q_3}$, $f_1 = 0$, $m_3 = p_3 = u$, $q_3 = v + 1$, $m_3 = 1 = q_3$, $n_3 = p_3 = 0$, $f_1 = 0$, $F_1 = 1$, replacing e_{P_3} and f_{q_3} by $1 - e_{q_3}$ and $1 - f_{q_3}$ and taking c tending to zero, we obtain the result by Singh[3]. equation (21).

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