BENDING OF AELOTROPIC BLOCKS-III

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The problem of bending of aelotropic circular blocks part I and II into ellipsoidal and paraboloidal shell have been discussed by the author[1,2] on the lines of Green & Adkins[3]. In this paper the problem of bending of an aelotropic circular block into hyperboloidal shell has been considered and its solution has been obtained in terms of a general strain energy function for compressible and incompressible materials.

The problems of bending of aelotropic circular blocks into ellipsoidal and paraboloidal shells have been considered by the author[1,2] earlier. The notation and formulae used are of Green and Adkins[3]. In this paper, the problem of bending of an aelotropic circular block into hyperboloidal shell has been considered on the lines of the earlier papers.

Suppose that a circular block is bounded, in the undeformed state, by the planes $x_3 = a_1$, $x_3 = a_2$ $(a_2 > a_1)$ and the cylinder $x_1^2 + x_2^2 = a^2$. The block is bent symmetrically about the x_3-axis into part of a hyperboloidal shell, whose inner and outer boundaries are obtained by revolving the confocal hyperbolas

$$x_3 = c \cos \xi_i \cosh \eta, \quad x_1 = c \sin \xi_i \sinh \eta, \quad i = 1,2 \tag{1}$$

about the x_3-axis, and the edge $\eta = \alpha$. Let y_i-axis coincide with the x_i-axis, and the curvilinear coordinates θ^i in the deformed state be a system of orthogonal curvilinear coordinates (ξ, η, φ). Then, since the deformation is symmetric about the x_3-axis, the deformation could be taken as

$$\xi = f(x_3), \quad \eta = K (x_1^2 + x_2^2)^{1/2}, \quad \varphi = \tan^{-1} (x_2/x_1). \tag{2}$$

The strain components e_{ij} are given by

$$\begin{aligned} 2e_{11} &= 2e_{22} = K^2 c^2 (\cosh^2 \eta - \cos^2 \xi) - 1, \\ 2e_{33} &= f'^2 (x_3) (\cosh^2 \eta - \cos^2 \xi) - 1, \\ e_{12} &= e_{23} = e_{31} = 0. \end{aligned} \tag{3}$$

The non-vanishing stress tensor has components

$$T^{11} = f'^2 (x_3) \Big/ \sqrt{I_3} \left(\frac{\partial W}{\partial e_{33}}\right), \quad T^{22} = \eta^2 \; T^{33} = K^2 \Big/ \sqrt{I_3} \left(\frac{\partial W}{\partial e_{11}}\right) \tag{4}$$

where

$$I_3 = K^4 f'^2 (x_3) \; c^6 (\cosh^2 \eta - \cos^2 \xi)^3 \tag{5}$$

The equations of equilibrium could be solved by supposing η to be small [1,2]. These give

$$f'^2 (x_3) = (W + W_0)/c^2 \sin^2 \xi \left(\frac{\partial W}{\partial e_{33}}\right) \tag{6}$$

Then the non-vanishing physical components of stress are given by

$$\sigma_{11} = \sqrt{(W + W_0) \frac{\partial W}{\partial e_{33}} \Big/ K^2 c^2 \sin^2 \xi}, \quad \sigma_{22} = \sigma_{33} = \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)} \tag{7}$$

Incompressible Material :

In this case $I_3 = 1$. Hence from (5), we obtain

$$\frac{df}{dx_3} = 1/K^2 c^3 \sin^3 \xi,$$

which on integration gives

$$x_3 = \frac{K^2 c^3}{4} \left(\frac{\cos^3 \xi}{3} - 3 \cos \xi\right) + B, \tag{8}$$

where B is an arbitrary constant.

As the internal and external boundaries of the shell are given by $\xi = \xi_i$, $i = 1, 2$ respectively, which were initially the planes $x_3 = a_1$ and $x_3 = a_2$, the values of K and B are obtained from (8) by using these relations.

Compressible Material :

The equation of equilibrium in this case is

$$P = W + W_0 - 1/K^4 c^4 \sin^4 \xi \ \frac{\partial W}{\partial e_{33}} \tag{9}$$

The physical components of stress are given by

$$\sigma_{11} = W + W_0 ,$$

$$\sigma_{22} = \sigma_{33} = W + W_0 + K^2 c^2 \sin^2 \xi \ (\partial W / \partial e_{11}) - \frac{1}{K^4 \ c^4 \ \sin^4 \ \xi} \cdot \frac{\partial W}{\partial e_{33}} . \tag{10}$$

BOUNDARY CONDITIONS

In the case of compressible material, if $-R_i$, $i = 1, 2$ are the applied normal tractions on the inner and the outer surfaces of the shell respectively, we have $\sigma_{11} = -R_i$ when $\xi = \xi_i$. These on substitution in (7) give the values of the constants W_0 and K.

The distribution of tractions on the edge $\eta = \alpha$ per unit arc between φ and $\varphi + d\varphi$ give rise to a force F_1 and a couple of moment M_1 about the origin which are given by

$$F_1 = \alpha \int_{\xi_1}^{\xi_2} c^2 \sin^2 \xi \ \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)} \ d\xi ,$$

$$M_1 = -\alpha \int_{\xi_1}^{\xi_2} c^2 \sin^2 \xi \ (c \cos \xi) \ \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)} \ d\xi . \tag{11}$$

In the case of incompressible material, if the inner boundary $\xi = \xi_1$ of the shell is supposed free from tractions, we must have $\sigma_{11} = 0$ when $\xi = \xi_1$, which on substitution in (10) gives $W_0 = -W(\xi_1)$. On the outer surface $\xi = \xi_2$ we have to apply a normal traction R given by

$$R = \sigma_{11} (\xi_2) = W (\xi_2) - W (\xi_1).$$

On the edge $\eta = \alpha$, the distribution of tractions between φ and $\varphi + d\varphi$ give rise to a force F_2 and a couple M_2 about the origin as

$$F_2 = \alpha \int_{\xi_1}^{\xi_2} c^2 \sin^2 \xi \left[W + W_0 + K^2 c^2 \sin^2 \xi \ \frac{\partial W}{\partial e_{11}} - \frac{1}{K^4 c^4 \sin^4 \xi} \ \frac{\partial W}{\partial e_{33}} \right] d\xi$$

$$M_2 = - \int_{\xi_1}^{\xi_2} c^2 \sin^2 \xi \ (c \cos \xi) \left[W + W_0 + K^2 c^2 \sin^2 \xi \ \frac{\partial W}{\partial e_{11}} - \frac{1}{K^4 c^4 \sin^4 \xi} \ \frac{\partial W}{\partial e_{33}} \right] d\xi . \tag{12}$$

As in Parts I and II, in this problem also we require a force F and a couple M on the edge together with a normal traction R on the outer edge to bend the block into hyperboloidal shell.

REFERENCES

1. RAM RAO, S. & LAKSHMINARAYANA, G., *Def. Sci. J.*, **23** (1973), 11.
2. RAM RAO, S. & LAKSHMINARAYANA, G., *Def. Sci. J.*, **23** (1973), 115.
3. GREEN, A. E. & ADKINS, J. E., *Arch. Rat. Mech. Anal.*, **8** (1961), 9