QUASISTATIC RESPONSE OF AN INFINITE HOLLOW VISCOELASTIC CYLINDER UNDER TIME VARYING INTERNAL PRESSURE

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The quasistatic response of an infinite hollow viscoelastic cylinder in terms of radial displacement, strain and stress components is derived from the fundamentals corresponding to a time varying internal pressure by using Fourier transform method and as a specific example, the problem of an infinite hollow cylinder, made of ebonite and subject to constant internal pressure, has been solved analytically.

Analytical solutions for the quasistatic problems of viscoelastic cylinders have been presented earlier by Lee[1], Bland[2] and Christensen[3] using Laplace transform method. The solutions for these problems are all based on the elastic-viscoelastic correspondence principle.

The present paper deals with the solution expressed in the form of displacement, strain and stress components within an infinite hollow viscoelastic cylinder subject to time varying internal pressure. The Fourier transform method which has been used here to obtain the solution, is considered more convenient for obtaining the quantitative solution of the problems concerning viscoelastic materials with the properties expressed as complex function of frequency.

FIELD EQUATIONS

In polar cylindrical coordinates r, θ, z with axial symmetry the nontrival elastic equation of motion is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = 0 \tag{1}$$

where, $u = u(r, t)$ is the radial displacement. Using Fourier transform method, the equation is converted to

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} = 0 \tag{2}$$

where $\bar{u}=\bar{u}(r, w)$, w being transform parameter. Strain-displacement and stress-strain relation are given by

$$\bar{\epsilon}_{rr} = \frac{\partial \bar{u}}{\partial r}, \ \epsilon\theta\theta = \frac{\bar{u}}{r}, \ \bar{\epsilon}_{zz} = 0 \tag{3}$$

$$\bar{\sigma}_{rr} = 2\bar{\mu}\left(\frac{\partial \bar{u}}{\partial r} + \frac{\bar{\nu}}{1-2\bar{\nu}} . \bar{e}\right) \tag{4}$$

$$\bar{\sigma}\theta\theta = 2\bar{\mu}\left(\frac{\bar{u}}{r} + \frac{\bar{\nu}}{1-2\bar{\nu}} . \bar{e}\right) \tag{5}$$

$$\bar{\sigma}_{zz} = 2\bar{\mu}\left(\frac{\bar{\nu}}{1-2\bar{\nu}}\right) . \bar{e} \tag{6}$$

where

$\bar{\mu}$ = Transformed shear modulus = $\mu^* (iw)$

$\bar{\nu}$ = Transformed Poisson's ratio

$\bar{e} = \bar{\epsilon}_{rr} + \bar{\epsilon}\theta\theta$ = Transformed 1st invarient of strain

SOLUTIONS FOR DISPLACEMENT STRAIN AND STRESS COMPONENTS

The solution for $\bar{u}$ is obtained from (2) as

$$\bar{u} = \bar{C}\, r + \frac{\bar{D}}{r} \tag{7}$$

where the constants $\bar{C}$ and $\bar{D}$ are determined from the boundary conditions given by

$$\bar{\sigma}_{rr}(a, w) = \bar{p}(w) \text{ or } \bar{p}, \; \bar{\sigma}_{rr}(b, w) = 0 \tag{8}$$

'a' and 'b' are internal and external radius respectively. Using (8)

$$\bar{C} = -\left(\frac{a^2}{b^2 - a^2}\right)(1 - 2\bar{\nu})\frac{\bar{p}}{2\bar{\mu}} \tag{9}$$

$$\bar{D} = -\left(\frac{a^2}{b^2 - a^2}\right)\frac{\bar{p}}{2\bar{\mu}} \tag{10}$$

On using (9) and (10), equations (3), (4), (5) and (6) yield solutions in transformed form

$$\bar{u} = -\left(\frac{a^2}{b^2 - a^2}\right)\left(r + \frac{b^2}{r}\right)\left\{(1 - 2\bar{\nu})\frac{\bar{p}}{2\bar{\mu}}\right\} \tag{11}$$

$$\bar{\epsilon}_{rr} = -\left(\frac{a^2}{b^2 - a^2}\right)\left(1 - \frac{b^2}{r^2}\right)\left\{(1 - 2\bar{\nu})\frac{\bar{p}}{2\bar{\mu}}\right\} \tag{12}$$

$$\bar{\epsilon}_{\theta\theta} = -\left(\frac{a^2}{b^2 - a^2}\right)\left(1 + \frac{b^2}{r^2}\right)\left\{(1 - 2\bar{\nu})\frac{\bar{p}}{2\bar{\mu}}\right\} \tag{13}$$

$$\bar{\sigma}_{rr} = -\left(\frac{a^2}{b^2 - a^2}\right)\left(1 - \frac{b^2}{r^2}\right)\bar{p} \tag{14}$$

$$\bar{\sigma}_{\theta\theta} = -\left(\frac{a^2}{b^2 - a^2}\right)\left(1 + \frac{b^2}{r^2}\right)\bar{p} \tag{15}$$

$$\bar{\sigma}_{zz} = -\left(\frac{a^2}{b^2 - a^2}\right)2\bar{\nu}\,\bar{p} \tag{16}$$

The solutions for an ebonite cylinder subject to constant internal pressure 'p_0' can be obtained in following manner,

$$\bar{p}(w) = \int_0^\infty p_0 H(t)\, e^{-iwt}\partial t = \left(\frac{p_0}{w}\right), \quad \text{where} \quad H(t) = 1,\ t \geqslant 0 = 0\ ;\ t < 0$$

$$\bar{\mu} = \mu^*(iw) = K\,(iw)^{\nu}$$

determined by Lethersich and reported by Bland[2]

ν = constant Poisson's ratio

By using the above relations and the Fourier inversion formula along with the table of Fourier transforms[4], the time dependent solutions for displacement, strain and stress can be obtained as

$$u(r, t) = G_1\left(r + \frac{b^2}{r}\right)t^{\nu} \tag{17}$$

where

$$G_1 = \left(\frac{a^2}{b^2 - a^2}\right)\left(1 - 2\nu\right)\frac{p_0}{4\pi K}\sin\pi(1 + \nu)\,(-\nu)^{!}$$

$$(-\nu)^{!} = \text{Gamma function of } (-\nu)$$

Choosing $\rho = \dfrac{r}{a}$, $R = \dfrac{b}{a}$, the displacement and strain components can be expressed in non-dimensional form as

$$u^* (\rho, t) = \frac{u (r, t)}{C_1 \, a} = \left(\rho + \frac{R^2}{\rho} \right) t^{\nu} \tag{18}$$

Similarly strain components are

$$\epsilon_{rr}^* (\rho, t) = \frac{\epsilon_{rr} (r, t)}{C_1} = \left(1 - \frac{R^2}{\rho^2} \right) t^{\nu} \tag{19}$$

$$\epsilon_{\theta\theta}^* (\rho, t) = \frac{\epsilon_{\theta\theta} (r, t)}{C_1} = \left(1 + \frac{R^2}{\rho^2} \right) t^{\nu} \tag{20}$$

stress components are

$$\sigma_{rr} (r, t) = - \left(\frac{a^2}{b^2 - a^2} \right) \left(1 - \frac{b^2}{r^2} \right) p_0 \, H (t) \tag{21}$$

$$\sigma_{\theta\theta} (r, t) = - \left(\frac{a^2}{b^2 - a^2} \right) \left(1 + \frac{b^2}{r^2} \right) p_0 \, H (t) \tag{22}$$

$$\sigma_{zz} (r, t) = - \left(\frac{a^2}{b^2 - a^2} \right) 2 \, \nu \, p_0 \, H (t) \tag{23}$$

NUMERICAL COMPUTATIONS

The geometry of the cylinder is such that $\frac{b}{a} = R = 3$. $\nu = 0.014$. Since the response is sufficiently slow with respect to time calculation of radial displacement $u^*(\rho, t)$, strain components $\epsilon^*_{rr} (\rho, t)$ & $\epsilon^*_{\theta\theta} (\rho, t)$ are performed at different times between 10^5 sec (approximately one day after loading) and 10^7 sec, and at different radial locations. The calculated results are tabulated in Tables 1, and 2.

TABLE 1

RADIAL DISPLACEMENT U^* CALCULATED AT DIFFERENT TIMES AND AT DIFFERENT RADIAL POINTS

t (sec)	$\rho=1$	$\rho=1\cdot5$	$\rho=2$	$\rho=2\cdot5$	$\rho=3\cdot0$
10^5	11·17	8·2	7·6	5·38	4·675
$10^{5\cdot5}$	11·94	8·36	7·75	5·5	4·75
10^6	12·13	8·5	7·9	5·6	4·85
$10^{6\cdot5}$	12·32	8·65	8·05	5·67	4·925
10^7	12·52	8·76	8·15	5·76	5·00

TABLE 2

STRAIN COMPONENTS, ϵ^*_{rr} & $\epsilon^*_{\theta\theta}$ CALCULATED AT DIFFERENT TIMES AND AT DIFFERENT RADIAL POINTS

t (sec)	$\rho=1$		$\rho=2$		$\rho=3$	
	ϵ^*_{rr}	$\epsilon^*_{\theta\theta}$	ϵ^*_{rr}	$\epsilon^*_{\theta\theta}$	ϵ^*_{rr}	$\epsilon^*_{\theta\theta}$
10^5	—9·35	11·17	—1·46	3·8	0	2·35
$10^{5\cdot5}$	—9·5	11·94	—1·49	3·875	0	2·38
10^6	—9·7	12·13	—1·51	3·95	0	2·43
$10^{6\cdot5}$	—9·85	12·32	—1·545	4·02	0	2·47
10^7	—10·0	12·52	—1·562	4·075	0	2·51

In the present problem which relates to the viscoelastic medium, it is seen that even though the loading is time-independent the response is time-dependent. This can be attributed to the dependence of the properties of viscoelastic material on frequency on time. This time dependent strain or displacement is called the creep. It is further noticed that the variation of strain component with time at a particular radial location for the viscoelastic cylinder under constant internal pressure bears an analogy to one suggested by Goldhoff[5] for C_r —Mo—V steel at 538°C under constant stress.

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