ON A FUNCTIONAL EQUATION IN TWO AND THREE VARIABLES WITH MEASURES OF KIND β IN INFORMATION THEORY

RAMINDER SINGH SONI

University of Delhi, Delhi

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Starting from a functional equation of two variables containing a parameter β, the information function of kind β given have been obtained. The paper contains a characterization and properties of this measure. Also the study has been extended to the information improvement of the prediction probability distribution $(q_1, q_2, \ldots q_n)$ revised as $(r_1, r_2 \ldots r_n)$ on the basis of realization $(p_1, p_2, \ldots p_n)$.

Let $P = (p_1, p_2, \ldots, p_n)$ $p_i \geqslant 0$, $\sum_{i=1}^{n} p_i = 1$ be a probability distribution of a set of n events on the basis of an experiment E whose true (prediction) probability distribution is $Q = (q_1, q_2, \ldots, q_n)$, $q_i \geqslant 0$, $\sum_{i=1}^{n} q_i = 1$, then Kullback's[1] measure of information that P provides about Q is

$$I(P; Q) = \sum_{i=1}^{n} p_i \log (p_i q_i^{-1}). \tag{1}$$

Again let $R = (r_1, r_2, \ldots, r_n)$, $r_i \geqslant 0$, $\sum_{i=1}^{n} r_i = 1$ be a revised probability distribution of a original probability distribution $Q = (q_1, q_2, \ldots, q_n)$, $q_i \geqslant 0$, $\sum_{i=1}^{n} q_i = 1$ on the basis of realization $P = (p_1, p_2, \ldots, p_n)$, $p_i \geqslant 0$, $\sum_{i=1}^{n} p_i = 1$ obtained from the experiment E. Then the information improvement[2] is given by

$$I(P; Q; R) = \sum_{i=1}^{n} p_i \log (r_i q_i^{-1}). \tag{2}$$

In this paper we shall take the functional equation in two and three variables, involving a parameter β which under suitable boundary conditions would give the new measures named as the 'information' and 'information-improvement' of kind β[3]. The paper contains the characterizations of these new measures and some of their properties.

Sharma and Autar[4] have studied earlier[5,6] some other generalizations of (1) and (2) differently.

CHARACTERIZATION OF INFORMATION OF KIND β

We shall first take generalization of (1).

Let the information of kind β of P with respect to Q denoted by L_n be taken to satisfy the following postulates :

Postulate I

$$L_n \begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} = L_{n-1} \begin{pmatrix} P_2, p_3, \ldots, p_n \\ Q_2, q_3, \ldots, q_n \end{pmatrix} + P_2^{\beta} \, Q_2^{1-\beta} \, L_2 \begin{pmatrix} p_1/P_2, p_2/P_2 \\ q_1/Q_2, q_2/Q_2 \end{pmatrix},$$

where $P_2 = p_1 + p_2, Q_2 = q_1 + q_2, P_2, Q_2 > 0$.

Postulate II

$L_3 \begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix}$ is a symmetric function such that for any permutation of p's there is the same permutation of q's.

Postulate III

$$L_2 \begin{pmatrix} \beta^{1/\beta}, 1-\beta^{1/\beta} \\ 1 \quad , \quad 0 \end{pmatrix} = 1. \quad (0 < \beta < 1).$$

Note 1: The postulate I is a recursivity of information of kind β and it is a relation corresponding to additivity. Also postulate III can be considered as a normalization property and here it includes the parameter β for the quantities were $0 < \beta < 1$.

Note 2 : If we take

$$f(x, y) = L_2 \begin{pmatrix} x, 1-x \\ y, 1-y \end{pmatrix}, \tag{3}$$

then with the help of postulates I and II, L_n ($n \geqslant 3$) can be expressed in terms of the single function $f(x, y)$.

The postulate I at once gives the elegant form

$$L_n = \sum_{i=2}^{n} P_i^{\beta} Q_i^{1-\beta} \, . \, f(p_i/P_i, q_i/Q_i), \tag{4}$$

where $P_i = p_1 + p_2 + \ldots + p_i$, $Q_i = q_1 + q_2 + \ldots + q_i$, $i = 1, 2, \ldots, n$ with $P_n = Q_n = 1$.

When recursivity and symmetry of information of kind β are applied[7] to L_3, we obtain the functional equation

$$\begin{aligned} f(x, y) + (1-x)^{\beta} (1-y)^{1-\beta} f[u/(1-x), v/(1-y)] \\ = f(u, v) + (1-u)^{\beta} (1-v)^{1-\beta} f[x/(1-u), y/(1-v)], \\ \text{for } x, y, u, v \in [0, 1] \text{ with } x+u, y+v \in [0, 1]. \end{aligned} \tag{5}$$

If we put $y = x$ and $v = u$ and take $f(x, x) = f(x)$, (5) reduces to Kendall's[8] functional equation of information function.

We now adopt the following definition.

Definition : A real-valued solution $f(x, y)$ of (5) defined on $[0, 1] \times [0, 1]$ is said to be an information function of kind $\beta \in (0, 1)$, if it satisfies the following boundary conditions

$$f(0, 0) = f(1, 1), \tag{6}$$

$$f(\beta^{1/\beta}, 1) = f(1-\beta^{1/\beta}, 0) = 1, \tag{7}$$

Again if $f(x, y)$ is an information function of kind β then the information of kind β that P provides about Q is given by (4).

It is shown below that $f(x, y) = f(1-x, 1-y)$ and $f(0, 0) = 0$. Thus (7) follows by putting $x = \beta^{1/\beta}$ and $y = 1$ in (5).

We now give characterizations of information function and information of kind $\beta (\neq 1)$ in theorems I and II respectively.

Theorem I

The only solution $f(x, y)$ of (5) satisfying the additional conditions (6), and (7) is given by

$$f(x, y) = [1 - x^{\beta} y^{1-\beta} - (1-x)^{\beta} (1-y)^{1-\beta}] (1-\beta)^{-1}, \tag{8}$$

for $(x, y) \in [0, 1] \times [0, 1] \sim (0, 1) \, U \, (1, 0)$; $\beta \in (0, 1)$,

(We shall take $0^{\alpha} = 0$ $(\alpha \neq 0)$.

Proof: Taking $x = y = 0$ in (5), $f(0, 0) = 0$ and hence from (6), we get

$$f(1, 1) = f(0, 0) = 0. \tag{9}$$

Now replacing u, v in (5) by $1 - x$ and $1 - y$ respectively and applying (9), we have

$$f(x, y) = f(1-x, 1-y) \text{ for } x, y \in [0, 1]. \tag{10}$$

Let p_1, p_2, q_1, q_2 be four arbitrary numbers belonging to (0, 1). Setting $p_1 = 1-x$, $p_2 = u(1-x)^{-1}$, $q_1 = 1-y$, $q_2 = v(1-y)^{-1}$ in (5) and using (10), we have

$$f(p_1, q_1) + p_1^{\beta} q_1^{1-\beta} f(p_2, q_2) = f(p_1 p_2, q_1 q_2) + (1-p_1 p_2)^{\beta} (1-q_1 q_2)^{1-\beta} \cdot f(1-p_1/(1-p_1 p_2), 1-q_1/(1-q_1 q_2)) \tag{11}$$

for $p_1, q_1 \in [0, 1]$, $p_2, q_2 \in [0, 1]$ such that $p_1 p_2 \neq 1$ and $q_1 q_2 \neq 1$.

Take the function

$$F(p_1, p_2 ; q_1, q_2) = f(p_1, q_1) + [p_1^{\beta} q_1^{1-\beta} + (1-p_1)^{\beta} (1-q_1)^{1-\beta}] f(p_2, q_2),$$

where $p_1, p_2, q_1, q_2 \in (0, 1)$. (12)

We shall show that $F(p_1, p_2 ; q_1, q_2)$ is symmetric, *i.e.*,

$$F(p_1, p_2 ; q_1, q_2) = F(p_2, p_1 ; q_2, q_1). \tag{13}$$

From (11) and (12) we have

$$F(p_1, p_2 ; q_1, q_2) = f(p_1 p_2, q_1 q_2) + (1-p_1 p_2)^{\beta} (1-q_1 q_2)^{1-\beta} [f(1-p_1/(1-p_1 p_2), 1-q_1/(1-q_1 q_2)) + (1-p_1/(1-p_1 p_2))^{\beta} (1-q_1/(1-q_1 q_2))^{1-\beta} f(p_2, q_2)] \tag{14}$$

Setting $p_1^* = (1-p_1)(1-p_1 p_2)^{-1}$, $q_1^* = (1-q_1)(1-q_1 q_2)^{-1}$, we have

$$A(p_1, p_2 ; q_1, q_2) = f(1-p_1/(1-p_1 p_2), 1-q_1/(1-q_1 q_2)) + (1-p_1/(1-p_1 p_2))^{\beta} \cdot (1-q_1/(1-q_1 q_2))^{1-\beta} f(p_2, q_2)$$

$$= f(p_1^*, q_1^*) + p_1^{*\beta} q_1^{*(1-\beta)} f(p_2, q_2)$$

using (10) and (11) we get

$$= f\left(\frac{1-p_2}{1-p_1 p_2}, \frac{1-q_2}{1-q_1 q_2}\right) + \left(\frac{1-p_2}{1-p_1 p_2}\right)^{\beta} \left(\frac{1-q_2}{1-q_1 q_2}\right)^{1-\beta} f(p_1, q_1)$$

$$= A(p_2, p_1 ; q_2, q_1).$$

So it follows from (12) that

$$F(p_1, p_2 ; q_1, q_2) - F(p_2, p_1 ; q_2, q_1) = 0, \text{ proving (13).}$$

Next putting $p_2 = \beta^{1/\beta}$, $q_2 = 1$ or $p_2 = 1 - \beta^{1/\beta}$, $q_2 = 0$ in (13) and using the definition of $F(p_1, p_2 ; q_1, q_2)$ and (7) we get

$$0 = f(p_1, q_1) + [p_1^{\beta} q_1^{1-\beta} + (1-p_1)^{\beta} (1-q_1)^{1-\beta}] - 1 - \beta f(p_1, q_1).$$

From this it follows that

$$f(p_1, q_1) = [1 - p_1^{\beta} q_1^{1-\beta} - (1-p_1)^{\beta} (1-q_1)^{1-\beta}] (1-\beta)^{-1}, \text{ for all } p_1, q_1 \in (0, 1)$$

Since $f(0,0)=f(1,1)=0$, the result is true when $p_1=q_1=0$ or $p_1=q_1=1$.

Theorem II

Let $P=(p_1, p_2, \ldots, p_n)$, $Q=(q_1, q_2, \ldots, q_n)$ be two complete probability distributions then the information of kind β derived from the information function of kind β is in general given by (4) and if the information function $f(x, y)$ of kind β is obtained in theorem I, then the corresponding information of kind β is

$$\overset{\beta}{H_n}(P;Q)=\left[1-\sum_{i=1}^{n} p_i^{\beta} q_i^{1-\beta}\right](1-\beta)^{-1},\ \beta\neq 1,\ \beta>0. \tag{15}$$

Proof: Substituting the expression for $f(x, y)$ from (8) in (4) we have

$$\overset{\beta}{H_n}(P;Q)=(1-\beta)^{-1}\sum_{i=2}^{n}\left[P_i^{\beta} Q_i^{1-\beta}-P_{i-1}^{\beta} Q_{i-1}^{1-\beta}-p_i^{\beta} q_i^{1-\beta}\right]$$

$$=(1-\beta)^{-1}\left[P_n^{\beta} Q_n^{1-\beta}-P_1^{\beta} Q_1^{1-\beta}-\sum_{i=2}^{n} p_i^{\beta} q_i^{1-\beta}\right]$$

$$=(1-\beta)^{-1}\left[1-\sum_{i=1}^{n} p_i^{\beta} q_i^{1-\beta}\right],$$

clearly, it is (15).

PROPERTIES OF INFORMATION OF KIND β

Several properties like non-negativity, symmetry, null-information and expansibility[9] can be easily derived for $H_n(P;Q)$.

We mention below some other properties for $H^{\beta}_n(P;Q)$.

Strongly Additive property:

$$\overset{\beta}{H_{mn}}\begin{pmatrix} p_1{*}P_1', p_2{*}P_2', \ldots, p_n{*}P_n' \\ q_1{*}Q_1', q_2{*}Q_2', \ldots, q_n{*}Q_n'\end{pmatrix}=\overset{\beta}{H_n}\begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n\end{pmatrix}+$$

$$+\sum_{i=1}^{n} p_i^{\beta} q_i^{1-\beta}\ \overset{\beta}{H_m}\begin{pmatrix} P_i' \\ Q_i'\end{pmatrix}. \tag{16}$$

where $P_i'=(p_{1i}, p_{2i}, \ldots, p_{mi})$, $Q_i'=(q_{1i}, q_{2i}, \ldots, q_{mi})$,

$p_1{*}P_1'=(p_1 p_{11}, p_1 p_{21}, \ldots, p_1 p_{m1})$ etc. and $\sum_{j=1}^{m} p_{ji}=1$, $\sum_{j=1}^{m} q_{ji}=1$

for all $i=1, 2, \ldots n$.

An interesting special case of (16) is given below :

$$\overset{\beta}{H_{mn}}\begin{pmatrix} P{*}P' \\ Q{*}Q'\end{pmatrix}=H^{\beta}_n\begin{pmatrix} P \\ Q\end{pmatrix}+H^{\beta}_m\begin{pmatrix} P' \\ Q'\end{pmatrix}+(\beta-1)\overset{\beta}{H_n}\begin{pmatrix} P \\ Q\end{pmatrix}\overset{\beta}{H_m}\begin{pmatrix} P' \\ Q'\end{pmatrix} \tag{17}$$

where $P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n)$, $P' = (P_1, P_2, \ldots, P_m)$,

$$Q' = (Q_1, Q_2, \ldots, Q_m) \text{ and } \sum_{j=1}^{m} P_j = 1, \sum_{j=1}^{m} Q_j = 1.$$

When $\beta \to 1$, the last term in (17) vanishes and we have the well known additivity property of Kullback's information.

When $\beta \neq 1$ we have

$$\overset{\beta}{H}_{mn}\begin{pmatrix} P^* P' \\ Q^* Q' \end{pmatrix} < H^{\beta}_n \begin{pmatrix} P \\ Q \end{pmatrix} + H^{\beta}_m \begin{pmatrix} P' \\ Q' \end{pmatrix} \text{ for } \beta < 1.$$

ii Recursive-property

$$\overset{\beta}{H}_n \begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} - H^{\beta}_{n-1} \begin{pmatrix} p_1 + p_2, p_3, \ldots, p_n \\ q_1 + q_2, q_3, \ldots, q_n \end{pmatrix}$$

$$= (p_1 + p_2)^{\beta} (q_1 + q_2)^{1-\beta} \overset{\beta}{H}_2 \begin{pmatrix} \dfrac{p_1}{p_1 + p_2}, & \dfrac{p_2}{p_1 + p_2} \\ \dfrac{q_1}{q_1 + q_2}, & \dfrac{q_2}{q_1 + q_2} \end{pmatrix},$$

with $p_1 + p_2, q_1 + q_2 > 0.$

iii

$$\overset{\beta}{H}_m \begin{pmatrix} \sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{mi} \\ q_1, \ldots\ldots\ldots, q_m \end{pmatrix} \leqslant \sum_{i=1}^{n} p_i \overset{\beta}{H}_m \begin{pmatrix} q_{1i}, \ldots, q_{mi} \\ q_1, \ldots, q_m \end{pmatrix} \text{ where } \sum_{j=1}^{m} q_{ji} = 1$$

Proof
$$\overset{\beta}{H}_m \begin{pmatrix} \sum_{i=1}^{n} p_i q_{1i}, \ldots \sum_{i=1}^{n} p_i q_{mi} \\ q_1, \ldots\ldots\ldots, q_m \end{pmatrix} = (1-\beta)^{-1} \left[1 - \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i q_{ji} \right)^{\beta} q_j^{1-\beta} \right]$$

$$\leqslant (1-\beta)^{-1} \left[\sum_{i=1}^{n} p_i (1 - \sum_{j=1}^{m} q_{ji}^{\beta} q_j^{1-\beta} \right.$$

$$= \sum_{i=1}^{n} p_i H_m{}^{\beta} \begin{pmatrix} q_{1i}, \ldots, q_{mi} \\ q_1, \ldots, q_m \end{pmatrix},$$

because[10]

$$\left(\sum_{i=1}^{n} p_i q_{ji} \right)^{\beta} \geqslant \sum_{i=1}^{n} p_i q_{ji}{}^{\beta} \text{ if } \beta < 1 \quad (10)$$

CHARACTERIZATION OF INFORMATION IMPROVEMENT OF KIND β

In this section we shall study a generalization of (2). Let the information improvement of kind β of the probability distributions P, Q and R be denoted by U_n be taken to satisfy the following postulates

Postulate IV

$$U_n\begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \\ r_1, r_2, \ldots, r_n \end{pmatrix} = U_{n-1}\begin{pmatrix} P_2, p_3, \ldots, p_n \\ Q_2, q_3, \ldots, q_n \\ R_2, r_3, \ldots, r_n \end{pmatrix} + P_2\, Q_2^{1-\beta}\, R_2^{\beta-1}\, U_2\begin{pmatrix} p_1/P_2, p_2/P_2 \\ q_1/Q_2, q_2/Q_2 \\ r_1/R_2, r_2/R_2 \end{pmatrix}$$

where $P_2 = p_1 + p_2$, $Q_2 = q_1 + q_2$, $R_2 = r_1 + r_2$, $P_2, Q_2, R_2 > 0$.

Postulate V

$U_3\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \\ r_1, r_2, r_3 \end{pmatrix}$ is a symmetric function such that for any permutation of p's there are the same permutations of q's and r's.

Postulate VI

$$U_2\begin{pmatrix} \beta, & 1-\beta \\ 1, & 0 \\ 1, & 0 \end{pmatrix} = 1. \quad (0 < \beta < 1).$$

Note IV : The postulate IV is a recursivity of information improvement of kind β it and it is a relation corresponding to additivity. Also postulate VI can be considered as a normalization property and here it includes the parameter β for the quantities where $0 < \beta < 1$.

Note V : If we take

$$f(x, y, z) = U_2\begin{pmatrix} x, 1-x \\ y, 1-y \\ z, 1-z \end{pmatrix}, \tag{18}$$

then with the help of postulates IV and V, U_n $(n > 3)$ can be expressed in terms of the single function $f(x, y, z)$.

The postulate IV, at once gives the elegant form

$$U_n = \sum_{i=2}^{n} P_i\, Q_i^{1-\beta}\, R_i^{\beta-1}\, f(p_i/P_i, q_i/Q_i, r_i/R_i), \tag{19}$$

where $P_i = p_1 + p_2 + \ldots + p_i$, $Q_i = q_1 + q_2 + \ldots + q_i$, $R_i = r_1 + r_2 + \ldots + r_i$;
$i = 1, 2, \ldots, n$ with $P_n = Q_n = R_n = 1$.

When recursivity and symmetry of information improvement of kind β are applied to U_3, we obtain the functional equation

$$\begin{aligned} f(x, y, z) &+ (1-x)(1-y)^{1-\beta}(1-z)^{\beta-1} f(u/1-x, v/1-y, w/1-z) \\ &= f(u, v, w) + (1-u)(1-v)^{1-\beta}(1-w)^{\beta-1} f(x/1-u, y/1-v, z/1-w) \end{aligned} \tag{20}$$

for $x, y, z, u, v, w \in [0, 1]$ with $x + u, y + v, z + w \in [0, 1]$.

If we put $z = x$ and $w = u$ and take $f(x, y, z) = f(x, y)$, (20) reduces to (5), the functional equation for information function of kind β and again if we put $z = y = x$ and $w = v = u$ and take $f(x, x, x) = f(x)$, (20) reduces to Kendall's[8] functional equation for information function.

We now adopt the following definition.

Definition : A real-valued solution $f(x, y, z)$ of (20) defined on $[0, 1] \times [0, 1] \times [0, 1]$ is said to be an information improvement function of kind $\beta \in (0, 1)$, if it satisfies the following boundary conditions

$$f(0, 0, 0) = f(1, 1, 1), \tag{21}$$

$$f(\beta, 1, 1) = f(1-\beta, 0, 0) = 1, \tag{22}$$

for a given β such that $0 < \beta < 1$.

Again if $f(x, y, z)$ is an information improvement function of kind β $(\neq 1)$ then the information improvement of kind β $(\neq 1)$ is given by (19).

Note VI : It is shown below that $f(x,y,z) = f(1-x, 1-x, 1-z)$ and that $f(o,o,o) = o$. Now (22) follows by putting $x = \beta$, $y = 1$, $z = 1$ in (20).

We now give characterizations of information improvement function and information improvement of kind β $(\neq 1)$ in theorems III and IV respectively.

THEOREM III The only solution $f(x, y, z)$ of (20) satisfying the additional conditions (21) and (22) is given by

$$f(x, y, z) = \left[1 - xy^{1-\beta} z^{\beta-1} - (1-x)(1-y)^{1-\beta}(1-z)^{\beta-1}\right](1-\beta)^{-1} \tag{23}$$

for $(x,y,z) \in [0, 1] \times [0\ 1] \times [0, 1] \sim (x, 1, 0)\ U\ (x,0,0)\ U\ (x,0,1)\ U\ (x,1\ 1)$ (We shall take $0^{\alpha} = 0$ $(\alpha \neq 0)$).

Proof : Taking $x = y = z = 0$ in (20), $(f\,0, 0, 0,) = 0$ and hence from (21) we get

$$f(1, 1, 1,) = f(0, 0, 0,) = 0. \tag{24}$$

Now replacing u, v, w, in (20) by $1-x$, $1-y$, and $1-z$ respectively and applying (24) we have

$$f(x, y, z) = f(1-x, 1-y, 1-z) \tag{25}$$

for $x, y, z \in [0, 1]$.

Let $p_1, p_2, q_1, q_2, r_1, r_2$, be six arbitrary numbers from the open interval $(o, 1)$. Setting $p_1, = 1-x$, $p_2 = u(1-x)^{-1}$, $q_1 = 1-y$, $q_2 = v(1-y)^{-1}$, $r_1 = 1-z$, $r_2 = w(1-z)^{-1}$ in (20) and using (25), we have

$$f(p_1, q_1, r_1) + p_1 q_1^{1-\beta} r_1^{\beta-1} f(p_2, q_2, r_2) + (1-p_1 p_2)(1-q_1 q_2)^{1-\beta}(1-r_1 r_2)^{\beta-1}$$
$$\cdot f(1-p_1/(1-p_1 p_2), 1-q_1/(1-q_1 q_2), 1-r_1/(1-r_1 r_2)) \tag{26}$$

for $p_1, q_1, r_1 \in (0, 1)$, $p_2, q_2, r_2 \in [0, 1]$ such that $p_1 p_2 \neq 1$, $q_1 q_2 \neq 1$, $r_1 r_2 \neq 1$.

Consider the function

$$F(p_1, p_2; q_1, q_2; r_1, r_2) = f(p_1, q_1, r_1) + \left[p_1 q_1^{1-\beta} r_1^{\beta-1} + (1-p_1)(1-q_1)^{1-\beta}(1-r_1)^{\beta-1}\right]$$
$$f(p_2, q_2, r_2), \quad p_1, p_2, q_1, q_2, r_1, r_2 \in (0, 1). \tag{27}$$

We shall show that $F(p_1, p_2; q_1, q_2; r_1, r_2)$ is symmetric, i.e.,

$$F(p_1, p_2; q_1, q_2; r_1, r_2) = F(p_2, p_1; q_2, q_1; r_2, r_1). \tag{28}$$

Let us take $\beta \neq 1$ $(\beta > 0)$, then by (26),

$$F\ p_1, p_2; q_1, q_2; r_1, r_2) = f(p_1 p_2, q_1 q_2, r_1 r_2) + (1-p_1 p_2)(1-q_1 q_2)^{1-\beta}(1-r_1 r_2^{\beta-1}.$$

$$\left[f\left(1-p_1|(1 p_1 p_2), 1-q_1|(1-q_1 q_2), 1-r_1|(1-r_1 r_2)\right) + (1-p_1|(1-p_1 p_2))(1-q_1|(1-q_1 q_2)^{1-\beta}\right.$$

$$\cdot\ (1-r_1/(1-r_1 r_2))\ f\ (p_2, q_2, r_2)\Big] \tag{29}$$

Again setting $p^*_1 = (1-p_1)(1-p_1 p_2)^{-1}$, $q^*_1 = (1-q_1)(1-q_1 q_2)^{-1}$, $r^*_1 = (1-r_1)(1-r_1 r_2)^{-1}$,

$$\begin{aligned} A\,(p_1, p_2; q_1, q_2, r_1, r_2) = \\ &= f\,(1-p_1 \mid (1-p_1 p_2),\ 1-q_1 \mid (1-q_1 q_2),\ 1-r_1 \mid (r-r_1 r_2)) + \\ &+ (1-p_1 \mid 1-p_1 p_2)\,(1-q_1 \mid 1-q_1 q_2)^{1-\beta}\,(1-r_1 \mid 1-r_1 r_2).^{\beta-1}\, f\,(p_2\, q_2 r_2) \\ &= f\,(p^*_1, q^*_1, r^*_1) + p^*_1\, q^{*\,1-\beta}_1\, r^{*\,\beta-1}_1\ f\ (p_2, q_2, r_2) \\ &= f\,(1-p_2 \mid (1-p_1 p_2),\ 1-q_2 \mid (1-q_1 q_2),\ 1-r_2 \mid (1-r_1 r_2)) + \\ &+ (1-p_2 \mid 1-p_1 p_2)\,(1-q_2 \mid 1-q_1 q_2)^{1-\beta}\,(1-r_2 \mid 1-r_1 r_2)^{\beta-1}\, f\,(p_1, q_1, r_1) \\ &= A\,(p_2, p_1; q_2, q_1; r_2, r_1) \end{aligned}$$

Then it follows from (27) that

$$F\ (p_1, p_2; r_1, q_2;)\ q_1, r_2) - F\ (p_2, p_1; q_2, q_1; r_2, r_1) = 0, \quad \text{proving (28)}$$

Next putting $p_2 = \beta$, $q_2 = 1$, $r_2 = 1$ or $p_2 = 1-\beta$, $q_2 = 0$, $r_2 = 0$ in (28) and using the definition of $F\ (p_1, p_2; q_1, q_2; r_1; r_2)$ and (22) we get

$$0 = f\,(p_1, q_1, r_1) + \Big[\ p_1\, q_1^{1-\beta}\, r_1^{\beta-1} + (1-p_1)\,(1-q_1)^{1-\beta}\,(1-r_1)^{\beta-1}\Big] - 1 - \beta f\,(p_1, q_1, r_1)$$

From this it follows that

$$f\,(p_1, q_1, r_1) = \Big[1 - p_1\, q_1^{1-\beta}\, r_1^{\beta-1} - (1-q_1)\,(1-q_1)^{1-\beta}\,(1-r_1)^{\beta-1}\Big]\,(1-\beta)^{-1}$$

for all $p_1,\ q_1, r_1 \in (0,\ 1)$.

The result is true even when $p_1 = q_1 = r_1 = 0$ or $p_1 = q_1 = r_1 = 1$.

THEOREM IV Let $P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n)$ and $R = (r_1, r_2, \ldots, r_n)$ be three complete probability distributions then the information of kind β derived from the information improvement function of kind β is in general given by (19) and if the information improvement function $f\,(x, y, z)$ of kind β is as obtained in theorm III then the corresponding information improvement of kind β is

$$\overset{\beta}{H}_n\ (P;\ Q;\ R) = \Big[1 - \sum_{i=1}^{n} p\ q_i^{1-\beta}\, r_i^{\beta-1}\Big]\,(1-\beta)^{-1},\ \beta \neq \beta > 0. \tag{30}$$

proof: Substituting the expression for $f\,(x, y, z,)$ from (23) in (19) we have

$$\begin{aligned} \overset{\beta}{H}_n\,(P;\,Q;\,R) &= (1-\beta)^{-1} \sum_{i=2}^{n} \Big[P_i\, Q_i^{1-\beta}\, R_i^{\beta-1}\ \ P_{i-1}\, Q_{i-1}^{1-\beta}\, R_{i-1}^{\beta-1} - p_i\, q_i^{1-\beta}\, r_i^{\beta-1}\Big] \\ &= (1-\beta)^{-1}\Big[P_n\, Q_n^{1-\beta}\, R_n^{\beta-1} - P_1\, Q_1^{1-\beta}\, R_1^{\beta-1} - \sum_{i=2}^{n} p_i\, q_i^{1-\beta}\, r_i^{\beta-1}\Big] \\ &= (1-\beta)^{-1}\Big[1 - \sum_{i-1}^{n} p_i\, q_i^{1-\beta}\, r_i^{\beta-1}\Big], \end{aligned}$$

clearly, which is (30).

PROPERTIES OF INFORMATION IMPROVEMENT OF KIND β

Several properties like symmetry, null-information, expansibility can be easily derived for $\overset{\beta}{H}_n\ (P;Q;R)$ We mention below, some other properties for $\overset{\beta}{H}_n\ (P;Q;R)$.

(*i*) *Strongly* (*Non-Commutative Additive Property*).

$$H^{\beta}_{mn}\begin{bmatrix} \overset{*}{p_1} P'_1, & \overset{*}{p_2} P'_2, \ldots, & \overset{*}{p_n} P'_n \\ \overset{*}{q_1} Q'_1, & \overset{*}{q_2} Q'_2, \ldots, & \overset{*}{q_n} Q'_n \\ \overset{*}{r_1} R'_1, & \overset{*}{r_2} R'_2, \ldots, & \overset{*}{r_n} R'_n \end{bmatrix} =$$

$$= H^{\beta}_{n}\begin{bmatrix} p_1, & p_2, & \ldots, & p_n \\ q_1, & q_2, & \ldots, & q_n \\ r_1, & r_2, & \ldots, & r_n \end{bmatrix} + \sum_{i=1}^{n} p_i\, q_i^{1-\beta}\, r_i^{\beta-1}\, H^{\beta}_{m}\begin{bmatrix} P'_i \\ Q'_i \\ R'_i \end{bmatrix} \tag{31}$$

where $P'_i = (p_{1i}, p_{2i}, \ldots, p_{mi})$, $Q'_i = (q_{1i}, q_{2i}, \ldots, q_{mi})$,

$R'_i = (r_{1i}, \ldots, r_{mi})$, $p_1 {*}\, P'_1 = (p_1\, p_{11}, p_1\, p_{21}, \ldots, p_1\, p_{mi})$ etc.,

and $\sum_{j=1}^{m} p_{ji}, \ldots, = 1$, $\sum_{j=1}^{m} q_{ji} = 1$ $\sum_{j=1}^{m} r_{ji} = 1$ for all $i = 1, 2, \ldots, n$.

An interesting special case of (31) is given below.

$$H^{\beta}_{mn}\begin{bmatrix} P^{*} P' \\ Q^{*} Q' \\ R^{*} R' \end{bmatrix} = H^{\beta}_{n}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} + H^{\beta}_{m}\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} + (\beta-1)\, H^{\beta}_{n}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} H^{\beta}_{m}\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} \tag{32}$$

where $P = (p_1, p_2, \ldots, p_n)$ and $P' = (P_1, P_2, \ldots, P_m)$ etc. and

$$\sum_{j=1}^{m} P_j = 1,\ \sum_{j=1}^{n} Q_j = 1 \text{ and } \sum_{j=1}^{m} R_j = 1.$$

When $\beta \to 1$, the last term in (32) vanishes and we have the known additive property of Theil's information-improvement.

When $\beta \neq 1$, we have

$$H^{\beta}_{mn}\begin{bmatrix} P^{*} P \\ Q^{*} Q' \\ R^{*} R' \end{bmatrix} \gtrless H^{\beta}_{n}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} + H^{\beta}_{m}\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix}$$

according as $(\beta-1)\, H^{\beta}_{n}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} H^{\beta}_{m}\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} \gtrless 0.$

(*ii*) *Recursive-Property*

$$H^{\beta}_{n}\begin{bmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \\ r_1, r_2, \ldots, r_n \end{bmatrix} - H_{n-1}\begin{bmatrix} p_1 + p_2, p_3, \ldots, p_n \\ q_1 + q_2, q_3, \ldots, q_n \\ r_1 + r_2, r_3, \ldots, r_n \end{bmatrix}$$

$$= (p_1 + p_2)\,(q_1 + q_2)^{1-\beta}\,(r_1 + r_2)^{\beta-1}\, H^{\beta}_{2}\begin{bmatrix} \dfrac{p_1}{p_1 + p_2}, & \dfrac{p_2}{p_1 + p_2} \\ \dfrac{q_1}{q_1 + q_2}, & \dfrac{q_2}{q_1 + q_2} \\ \dfrac{r_1}{r_1 + r_2}, & \dfrac{r_2}{r_1 + r_2} \end{bmatrix};\ (n > 3)$$

with $p_1 + p_2, q_1, + q_2, r_1 + r_2 > 0.$

ii

$$\overset{\beta}{H}_m \left[\begin{array}{c} \sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{mi} \\ q_1, \ldots, q_m \\ r_1, \ldots, r_m \end{array} \right] = \sum_{i=1}^{n} p_i \overset{\beta}{H}_m \left[\begin{array}{c} q_{1i}, \ldots, q_{mi} \\ q_1, \ldots, q_m \\ r_1, \ldots, r_m \end{array} \right]$$

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