ALTERNATE SERVERS WITH SET UP TIMES

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The paper analyzes a queueing model consisting of a single queue serviced by two servers alternately. A change over time is required when one server is replaced by another. In a busy period, only one server is servicing and when it empties the queue, the second server takes over and thus the two servers work alternately. The actual service starts after the change over time of the particular server under operation.

In the single server queueing process[1,3] it is assumed that there is only one server serving one or more queues. With the advent of complex mechanism in the electronic field two or more similar modules serving a single queue are highly likely but this process is essentially different from the multi-server queueing process considered by various workers in the sense that all the modules are not operated simultaneously but work alternately to process the jobs coming in a single queue. This increases the operability of the system for more time than using a single server unit. An additional advantage in studying such a system is that if two or more similar modules are provided for service mechanism and if they work alternately then there is a possibility of providing an efficient repairing schedule for the modules which are at rest. It is this motivation which leads to consider the alternating servers in queueing process.

Recently Scott[2] has considered the alternating servers working in shifts and has also indicated briefly the generalization of the model. In this paper we propose to study the alternate servers with initial set up time i.e., whenever server's switching over takes place some initial time is spent (which is random variable) in setting up the service mechanism. For example if two central processors are provided in a single computer unit and these two work alternately to process jobs coming in a single queue. The loading of the monitors of central processors can be viewed as initial set up times before processing the job.

The study of queueing problems has got immense use in organisation of service facilities in various sectors such as Ordinance Depots, refuling of aircrafts, computers and communication network in Defence

STATEMENT OF THE PROBLEM

We consider that customers are arriving in the system according to a Poission process with intensity λ and joining a single counter queue operated by two servers 1 and 2 who work alternately with independent service rates. Each server has some initiation time Q_i which is a random variable. The odd number busy periods are serviced by server 1, and the even No. busy periods serviced by server 2. We shall use the method of supplementary variables[4], for the formulation and solution of the problem. The probability density function (p.d.f.) of the service time of the servers be denoted by $S_1(\theta)$ and $S_2(\theta)$ with $\eta_1(\theta)d\theta$ and $\eta_2(\theta)\,d\theta$ as the conditional probability that the service completes between $(\theta, \theta + \alpha\theta)$ given that the service is not completed up to time θ.

Let $M(t)$ be the number of customers in the system and $N(t)$ the number of customers already served at time t and θ denote the time elapsed in a particular service. $P_1(n, t)$ and $P_2(n, t)$ are the probabilities that the system is empty followed by the switch over period of 1st and 2nd server respectively. $q_i(m, n, \theta, t)\,d\theta$ is the probability that the server ($i = 1, 2$) is busy with m units waiting, n units have been serviced, θ is the time elapsed in the current service at any time t. $Q_i(m, n, \theta, t)\,d\theta$ is the probability that m units are waiting, n units have been serviced in the previous busy periods with switch over time lies between θ and $\theta + d\theta$. The switch over (set up) time distribution are assumed to be $S_3(\theta)$ and $S_4(\theta)$ of server 1 and server 2 respectively $\phi_1(\theta)$ and $\phi_2(\theta)$ are the conditional probability densities that set up time will be completed between θ and $\theta + d\theta$ given that these have not been completed upto time θ.

Equations governing the system are as follows :

$$\frac{\partial q_1(m, n, \theta, t)}{\partial t} + \frac{\partial q_1(m, n, \theta, t)}{\partial \theta} + [\lambda + \eta_1(\theta)]\, q_1(m, n, \theta, t) = \lambda\, q_1(m-1, n, \theta, t) \quad \text{for } m \geqslant 1 \tag{1}$$

$$\frac{\partial q_2(m, n, \theta, t)}{\partial t} + \frac{\partial q_2(m, n, \theta, t)}{\partial \theta} + [\lambda + \eta_2(\theta)]\, q_2(m, n, \theta, t) = \lambda\, q_2(m-1, n, \theta, t) \quad \text{for } m \geqslant 1 \tag{2}$$

$$\frac{\partial Q_1(m,n,\theta,t)}{\partial t}+\frac{\partial Q_1(m,n,\theta,t)}{\partial \theta}+[\lambda+\phi_1(\theta)]\,Q_1(m,n,\theta,t)=\lambda\,Q_1(m-1,n,\theta,t) \quad \text{for } m>0 \tag{3}$$

$$\frac{\partial Q_2(m,n,\theta,t)}{\partial t}+\frac{\partial Q_2(m,n,\theta,t)}{\partial \theta}+[\lambda+\phi_2(\theta)]\,Q_2(m,n,\theta,t)=\lambda\,Q_2(m-1,n_1\theta,t) \quad \text{for } m>0 \tag{4}$$

$$\frac{d\,P_1(n,t)+\lambda\,P_1(n,t)}{dt}=\int_0^\infty q_2(1,n-1,\theta,t)\,\eta_2(\theta)\,d\theta \tag{5}$$

$$\frac{d\,P_2(n,t)+\lambda\,P_2(n,t)}{dt}=\int_0^\infty q_2(1,n-1,\theta,t)\,\eta_1(\theta)\,d\theta \tag{6}$$

Boundary Conditions

$$Q_1(m,n,0,t)=0 \quad \text{for } m>1 \tag{7}$$

$$Q_1(m,n,0,t)=\lambda\,P_1(n,t) \quad \text{for } m=1 \tag{8}$$

$$Q_2(m,n,0,t)=0 \quad \text{for } m>1 \tag{9}$$

$$Q_2(m,n,0,t)=\lambda\,P_2(n,t) \quad \text{for } m=1 \tag{10}$$

$$q_1(m,n,0,t)=\int_0^\infty q_1(m+1,n-1,\theta,t)\,\eta_1(\theta)\,d\theta+\int_0^\infty Q_1(m,n,\theta,t)\,\phi_1(\theta)\,d\theta \tag{11}$$

$$q_2(m,n,0,t)=\int_0^\infty q_2(m+1,n-1,\theta,t)\eta_2(\theta)\,d\theta+\int_0^\infty Q_2(m,n,\theta,t)\,\phi_2(\theta)\,d\theta \tag{12}$$

The range of values of n depends upon the initial condition and with the initial condition $Q_1(1,0,0,0)=\delta\,\theta$, n will take the values for the equations (1) to (4) as

(1) $n\geqslant 0$ (2) $n\geqslant 1$ (3) $n\geqslant 0$ (4) $n\geqslant 1$

SOLUTION OF THE PROBLEM

We define the generating functions as under :

$$F_1(\theta,x,y,t)=\sum_{m=1}^{\infty} x^m f_{1,m}(\theta,y,t)=\sum_{m=1}^{\infty} x^m \sum_{n=0}^{\infty} y^n\, q_1(m,n,\theta,t)$$

$$F_2(\theta,x,y,t)=\sum_{m=1}^{\infty} x^m f_{2,m}(\theta,y,t)=\sum_{m=1}^{\infty} x^m \sum_{n=1}^{\infty} y^n\, q_2(m,n,\theta,t)$$

$$G_1(\theta,x_1y,t)=\sum_{m=1}^{\infty} x^m g_{1,m}(\theta_1y_1t)=\sum_{m=1}^{\infty} x^m \sum_{n=0}^{\infty} y^n\, Q_1(m_1n_1\theta_1t)$$

$$G_2(\theta,x,y,t)=\sum_{m=1}^{\infty} x^m g_{2,m}(\theta,y,t)=\sum_{m=1}^{\infty} x^m \sum_{n=1}^{\infty} y^n\, Q_2(m,n,\theta,t)$$

$$H_1(y,t)=\sum_{n=0}^{\infty} y^n\, P_1(n,t)$$

$$H_2(y,t)=\sum_{n=0}^{\infty} y^n\, P_2(n,t)$$

Multiplying (1) to (4) by y^n and taking the summation over appropriate values of n and then multiplying the resultant equations by x^m and taking the summation over suitable values of m we get making use of the generating functions.

$$\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial \theta} + [\lambda (1-x) + \eta_1 (\theta)] F_1 = 0 \tag{13}$$

$$\frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial \theta} + [\lambda (1-x) + \eta_2 (\theta)] F_2 = 0 \tag{14}$$

$$\frac{\partial G_1}{\partial t} + \frac{\partial G_1}{\partial \theta} + [\lambda (1-x) + \phi_1 (\theta)] G_1 = 0 \tag{15}$$

$$\frac{\partial G_2}{\partial t} + \frac{\partial G_2}{\partial \theta} + [\lambda (1-x) + \phi_2 (\theta)] G_2 = 0 \tag{16}$$

Similarly multiplying (15) and (16) by y^n and taking the summation over n we get

$$\frac{d}{dt} H_1 (y, t) + \lambda H_1 (y, t) = y \int_0^\infty f_{2,1} (\theta, y, t) \eta_2 (\theta) \, d\theta \tag{17}$$

$$\frac{d}{dt} H_2 (y, t) + \lambda H_2 (y, t) = y \int_0^\infty f_{1,1} (\theta, y, t) \eta_1 (\theta) \, d\theta \tag{18}$$

We define the laplace transform of a real valued function $f(t)$ as

$$\bar{f} (s) = \int_0^\infty e^{-st} f(t) \, dt , \ Rl \ (s) > 0$$

Applying the laplace transform to (13) to (18) and making use of the initial condition

$$\bar{F}_1 (\theta, x, y, s) = {}^{-}{}_1 (0, x, y, s) \quad e^{-[\lambda (1-x) + s] \theta} \quad e^{-\int_0^\theta \eta_1 (\theta) \, d\theta} \tag{19}$$

$$\bar{F}_2 (\theta, x, y, s) = \bar{F}_2 (0, x, y, s) \quad e^{-[\lambda (1-x) + s] \theta} \quad e^{-\int_0^\theta \eta_2 (\theta) \, d\theta} \tag{20}$$

$$\bar{G}_1 (\theta, x, y, s) = [x + \bar{G}_1 (0, x, y, s)] \quad e^{-[\lambda (1-x) + s]} \quad e^{-\int_0^\theta \phi_1 (\theta) \, d\theta} \tag{21}$$

$$\bar{G}_2 (\theta, x, y, s) = \bar{G}_2 (0, x, y, s) \quad e^{-[\lambda (1-x) + s]} \quad e^{-\int_0^\theta \phi_2 (\theta) \, d\theta} \tag{22}$$

$$(\lambda + s) \bar{H}_1 (y, s) = y \int_0^\theta \bar{f}_{2,1} (\theta, y, s) \eta_1 (\theta) \, d\theta \tag{23}$$

$$(\lambda + s) \bar{H}_2 (y, s) = y \int_0^\theta \bar{f}_{1,1} (\theta, y, s) \eta_1 (\theta) \, d\theta \tag{24}$$

Now we evaluate the value of constants $\bar{F}_1 (0, x, y, s)$, $\bar{F}_2 (0, x, y, s)$, $\bar{G}_1 (0, x, y, s)$ and $\bar{G}_2 (0, x, y, s)$ as under. From the generating function, applying the boundary condition (11) and multiplying by x^m and taking the summation

$$\bar{F}_1 (0, x, y, s) = y/x \sum_{m=1}^\infty x^{m+1} \int_0^\infty f_{1, m+1} (\theta, y, s) \eta_1 (\theta) \, d\theta + \int_0^\infty \bar{G}_1 (\theta, x, y, s) \phi_1 (\theta) \, d\theta =$$

$$= y/x \int_0^\infty \bar{F}_1 (\theta, x, y, s) \eta_1 (\theta) \, d\theta - y \int_0^\infty \bar{f}_{1,1} (\theta, y, s) \eta_1 (\theta) \, d\theta + \int_0^\infty \bar{G}_1 (\theta, x, y, s) \phi_1 (\theta) \, d\theta \tag{25}$$

using eq. (24)

$$\bar{F}_1(0, x, y, s) = y/x \int_0^\infty \bar{F}_1(\theta, x, y, s)\, \eta_1(\theta)\, d\theta - (\lambda + s) H_2(y, s) + \int_0^\infty \bar{G}_1(\theta, x, y, s)\, d\theta \qquad (26)$$

substituting from eq. (19)

$$\bar{F}_1(0, x, y, s) = y/x \int_0^\theta \left[\bar{F}_1(\theta, x, y, s)\, e^{-[\lambda(1-x)+s]\theta}\, e^{-\int_0^\theta \eta_1(\theta)\, d\theta} \right] \eta_1(\theta)\, d\theta \qquad (27)$$

$$\bar{F}_1(0, x, y, s) = \int_0^\infty \frac{\bar{G}_1(\theta, x, y, s)\, \phi_1(\theta)\, d\theta - (\lambda + s)\, H_2(y, s)}{1 - y/x\, \bar{S}_1[\lambda(1-x)+s]} \qquad (28)$$

where $\qquad S_1(\theta) = \eta_1(\theta)\,.\, e^{-\int_0^\theta \eta_1(\theta)\, d\theta}$

In a similar way we can write

$$\bar{F}_2(0, x, y, s) = \int_0^\infty \frac{\bar{G}_2(\theta, x, y, s)\, \phi_2(\theta)\, d\theta - (\lambda + s)\, H_1(y, s)}{1 - y/x\, \bar{S}_2[\lambda(1-x)+s]} \qquad (29)$$

where $\qquad S_2(\theta) = \eta_2(\theta)\,.\, e^{-\int_0^\theta \eta_2(\theta)\, d\theta}$

Making use of boundary conditions (7) and (8) we can write

$$\bar{G}_1(0, x, y, s) = \lambda\, x\, H_1(y, s) \qquad (30)$$

Similarly making use of boundary conditions (9) and (10)

$$\bar{G}_2(0, x, y, s) = \lambda\, x\, H_2(y, s) \qquad (31)$$

Therefore, (21) and (22) will become

$$\bar{G}_1(\theta, x, y, s) = [x + \lambda\, x\, H_1(y, s)]\, e^{-[\lambda(1-x)+s]\theta}\, e^{-\int_0^\theta \phi_1(\theta)\, d\theta} \qquad (32)$$

$$\bar{G}_2(\theta, x, y, s) = \lambda\, x\, H_2(y, s)\, e^{-[\lambda(1-x)+s]\theta}\, e^{-\int_0^\theta \phi_2(\theta)\, d\theta} \qquad (33)$$

Now (28) and (29) will become

$$F_1(0, x, y, s) = \frac{[x + \lambda x\, H_1(y, s)]\, \bar{S}_3[\lambda(1-x)+s] - (\lambda + s)\, H_2(y, s)}{1 - y/x\, \bar{S}_1[\lambda(1-x)+s]} \qquad (34)$$

where $\qquad S_3(\theta) = \phi_1(\theta).\, e^{-\int_0^\theta \phi_1(\theta)\, d\theta}$

$$F_2(0, x, y, s) = \frac{\lambda\, x\, H_2(y, s)\, \bar{S}_4[\lambda(1-x)+s] - (\lambda + s)\, H_1(y, s)}{1 - y/x\, \bar{S}_2[\lambda(1-x)+s]} \qquad (35)$$

where $\qquad S_4(\theta) = \phi_2(\theta).\, e^{-\int_0^\theta \phi_2(\theta)\, d\theta}$

Denominators of (34) and (35) each has got a root inside the unit circle $1 \times 1 = 1$. Defining the roots as $\alpha_1(y, s)$ and $\alpha_2(y, s)$, the numerators of (34) and (35) should also vanish for $x = \alpha_1(y, s)$ and $\alpha_2(y, s)$ respectively

Therefore,
$$H_1(y,s) = \frac{[\lambda\, \alpha_1(y,s)\, \alpha_2(y,s)\, p_4(\alpha,s)]}{[(\lambda+s)^2 - \lambda^2 \alpha_1(y,s)\, \alpha_2(y,s)\, p_3(\alpha,s)\, p_4(\alpha,s)]} \tag{36}$$

$$H_2(y,s) = \frac{(\lambda+s)\, \alpha_1(y,s)}{[(\lambda+s)^2 - \lambda^2 \alpha_1(y,s)\, \alpha_2(y,s)\, p_3(\alpha,s)\, p_4(\alpha,s)]} \tag{37}$$

where $p_3(\alpha,s) = \bar{S}_3[\lambda\{1-\alpha_1(y,s)+s\}]\; p_4(\alpha_1 s) = \bar{S}_4[\lambda(1-\alpha_2)(y,s)+s]$

Defining the generating function of the queue length probabilities as

$$\pi(x,y,t) = H_1(y,t) + H_2(y,t) + \int_0^\infty F_1(x,y,\theta,t)\, d\theta + \int_0^\infty F_2(\theta,x,y,t)\, d\theta +$$
$$+ \int_0^\infty G_1(\theta,x,y,t)\, d\theta + \int_0^\infty G_2(\theta,x,y,t)\, d\theta \tag{38}$$

Substituting the values already obtained, laplace transform of the generating function of the joint distribution of the No. of units in the system and No. of services completed up to time t.

$$\pi(x,y,s) = H_1(y,s) + H_2(y,s) + \left\{\frac{(x+\lambda x H_1(y,s)\, S_3 \overline{\lambda}[1-x]+s) - (\lambda+s)\, H_2(y,s)}{1 - y/x\, S_1[\lambda(1-x)+s)}\right\} \cdot$$
$$\cdot \frac{1-S_1[\lambda(1-x)+s]}{\lambda(1-x)+s} + \left\{\frac{\lambda x H_2(y,s)\, \bar{S}_4[\lambda(1-x)+s] - (\lambda+s)\, H_1(y,s)}{1 - y/x\, \bar{S}_2[\lambda(1-x)+s]}\right\} \frac{1-\bar{S}_2[\lambda(1-x)+s]}{\lambda(1-x)+s} +$$
$$+ [x + \lambda x H_1(y,s)]\, \frac{1-\bar{S}_3[\lambda(1-x)+s]}{\lambda(1-x)+s} + \frac{\lambda x H_2(y,s)\,(1-\bar{S}_4[\lambda(1-x)+s]}{\lambda(1-x)+s} \tag{39}$$

when $S_3(s) = 1$ and $S_4(s) = 1$ i.e. when set up time is identically equal to zero, the result obtained here agrees with the results of Scott :

Laplace transform of the generating function of queue length distribution irrespective of the number of services completed is given by

$$\pi(x,1,s) = H_1(1,s) + H_2(1,s) + \left\{\frac{x + \lambda x H_1(1,s)\, \bar{S}_3[\lambda(1-x)+s] - (\lambda+s)\, H_2(1,s)}{1 - 1/x\, \bar{S}_1[\lambda(1-x)+s]}\right\} \cdot$$
$$\cdot \frac{1-\bar{S}_1[\lambda(1-x)+s]}{\lambda(1-x)+s} + \left\{\frac{\lambda x H_2(1,s)\, \bar{S}_4[\lambda(1-x)+s] - (\lambda+s) H_1(1,s)}{1 - 1/x\, \bar{S}_2[\lambda(1-x)+s]}\right\} \cdot$$
$$\cdot \frac{1-\bar{S}_2[\lambda(1-x)+s]}{\lambda(1-x)+s} + [x + \lambda x H_1(1,s)] \times \frac{(1-\bar{S}_3(\lambda(1-x)+s])}{\lambda(1-x)+s} +$$
$$+ \lambda x H_2(1,s) \left\{\frac{1-\bar{S}_4[\lambda(1-x)+s]}{\lambda(1-x)+s}\right\} \tag{40}$$

The steady state distribution of the queue length distribution can be obtained by taking the limit as $s \to 0$ $S\pi(x,1,s) = \pi(x,1)$

$$\pi(x,1) = H_1(1)\left[2 + x\left\{\frac{(\bar{S}_1\,\bar{S}_3 - 1)}{x - \bar{S}_1} + \frac{(\bar{S}_2\,\bar{S}_4 - 1)}{x - \bar{S}_2}\right\}\right] \tag{41}$$

where

$$H_1(1) = \left[\frac{1+\lambda\eta_3}{1-\lambda\eta_1} + \frac{1+\lambda\eta_4}{1-\lambda\eta_2}\right]^{-1}$$

which is the probability that no unit is present in the system. Then mean number of units in the system is given by

$$\text{Unit}\, x \to 1 \; \frac{d}{dx} \; \pi\,(x,\, 1)$$

$$= H_1\,(1) \left(\frac{-\lambda^2}{2} \right) \left[\frac{(\mu_1 + \mu_3 + \eta_1\,\eta_3)\;(1 - \lambda\,\eta_1) + \lambda\,\mu_1\,(\eta_1 + \eta_3)}{(1 - \lambda\,\eta_1)^2} + \right.$$

$$\left. + \frac{(\mu^1 + \mu_4 + \eta_2\,\eta_4)\;(1 - \lambda\,\eta_2) + \lambda\,\mu_2\,(\eta_2 + \eta_4)}{(1 - \lambda\,\eta_2)^2} \right] \qquad (42)$$

where $\eta_i = \int_0^\infty x\,S_i\,(x)\,dx$

$$\mu_i = \int_0^\infty x^2\,S_i\,(x)\,dx \quad (i = 1,\,4)$$

Similarly the system of (1) to (6) can be solved for any given initial conditions

CONCLUSION

In this paper it is assumed that set up time is initiated only when a customer arrives, during his tenure of servicing but it may be noted that set up time of a server can also be initiated when the previous server finishes his job. In that case empty state of the process will not reach, such a process will be dealt with in the subsequent paper.

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