NOTE ON SOLUTION FOR PLANE OSEEN FLOW PAST A SEMI-INFINITE PLATE

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The solution for viscous incompressible flow past a semi-infinite flat plate aligned with the flow are obtained under Oseen's approximation. The problem has been reduced to the solution of an integral equation which in turn reduces to the solution of a Reiman Hilbert boundary value problem. The boundary value problem has, ultimately, been solved by standard technique as sought by Gakhov.

One of the simplest classical problems in fluid mechanics is the flow of an incompressible viscous fluid past a semi-infinite plate aligned with the flow. Under the simplification of Oseen's approximation, the behaviour of the drag force near the trailing edge of the plate is to be treated carefully. For low Reynold's number the solution of the Oseen problem is carried out by a series in powers of the Reynold's number and the product of powers with logerithim of the Reynold's number. The treatment has been explained in the classical work of Piercy and winny.

Accurate results for the viscous flow past a flat plate under Oseen's approximation are needed for numerical studies of the same problem with full Navier stokes equation. While solving the same problem Olmstead[1] has obtained a singular integral equation involving the drag singularities along the plate. In general, standard Weiner-Hopf technique is employed for solving such equations. In order to avoid the difficulties in factorization in the Weiner-Hopf method, Olmstead[1] has considered a related integral equation which includes the obtained integral equations as a limiting case and used Fourier transform to obtain the desired result.

The author in this present note has considered the same problem as posed by Olmstead[1,2] and has reduced the solution of the integral equation to that of a Reiman—Hilbert boundary value problem and ultimately the boundary value problem has been solved by using Plemj's. Formula and Laplace transform.

FORMULATION OF THE PROBLEM OF SOLUTION

The viscous flow of an incompressible fluid past a flat plate under Oseen approximation can be determined from the solution, with appropriate boundary condition of the equation,

$$\vec{\nabla} \cdot \vec{V} = 0 \tag{1}$$

$$U \vec{i} \cdot \nabla \vec{V} = -\rho^{-1} \nabla \rho + \nu \nabla^2 \vec{V} + \rho^{-1} \Gamma \vec{i_1} \delta (x, y; -b, 0) \text{ in } D \tag{2}$$

where $\vec{V} = v(x, y)$ is the velocity vector,

P, the pressure, U the uniform velocity at infinity, ν the kinematic viscosity, ρ the density, $\vec{i_1}$ the unit vector in positive x-direction, $\delta(x, y; -b, 0)$ the Direct delta function and the region D is the external to the semi-infinite plate: $y = 0, 0 \leqslant x \angle \infty$. A line of concentrated horizontal force of strength $\gamma > 0$ has as its projection in (x, y) plane the point $(-b, 0)$, $b > 0$ directed along positive x-axis.

For the boundary condition on velocity, we require at infinity,

$$V \to \vec{i}\, U \tag{3}$$

while on the plate

$$V = 0 \tag{4}$$

It follows from Olmstead's analysis[2] that for $\Gamma = 0$ (which we consider for the present case), a solution of the Oseen equations (1) and (2) which satisfies the boundary condition (3) is given by,

$$V(x, y) = \vec{i_1} U + \int_0^\infty V_I(x, y; x_0, 0)\, \sigma(x_0)\, dx_0 \tag{5}$$

$$P(x, y) = \frac{\rho U}{2\pi} \int_0^{\infty} \frac{(x - x_0)\,\sigma(x_0)}{(x - x_0)^2 + y^2}\, dx_0 \tag{6}$$

where,

$$V_I(x, y, x_0, 0) = -\frac{1}{2\pi} \left\{ \nabla \log \left[(x - x_0)^2 + y^2 \right]^{\frac{1}{2}} + \text{Exp}\,[k(x - x_0)] \right.$$
$$\left. (\nabla - i_I k)\, K_0 \left\{ k\,[(x - x_0)^2 + y^2]^{\frac{1}{2}} \right\} \right\} \tag{7}$$

where $4k = \frac{2U}{\nu}$ the Reynolds No. K_0 is the modified Bessel function of second kind, the function $\sigma(x_0)$ is the distribution of shearing force along the plate (drag regularities). If we employ the boundary condition (4) on the plate we will arrive at an integral equation for $\sigma(x_0)$. By employing the boundary condition (4) on the expression for velocity as obtained in (5) the following integral equation for $\sigma(x_0)$ is obtained,

$$2\pi U = k \int_0^{\infty} Q\,[k(x - x_0)]\, \sigma(x_0)\, dx_0\,, \quad 0 \leqslant x < \infty \tag{8}$$

where

$$Q\left[k(x - x_0)\right] = \frac{1}{k(x - x_0)} - \left\{ \frac{x - x_0}{|x - x_0|} K_1(k|x - x_0|) + K_0(k|x - x_0|) \right\} \cdot \exp\,[k(x - x_0)] \tag{9}$$

For convenience the following variables are introduced:—

$$S = kx, \quad t = kx_0, \quad \bar{\sigma}(t) = \sigma(x_0)/2\pi U$$

The integral equation then becomes.

$$1 = \int_0^{\infty} Q(s - t)\, \bar{\sigma}(t)\, dt, \quad 0 \leqslant s < \infty \tag{10}$$

Our problem now is to solve the integral equation (10). In order to solve this we consider a sectionally analytic function $F(Z)$ which is of the order $\frac{1}{Z}$ or any higher order at infinity, and analytic outside the cut $(0, \infty)$ of the complex Z plane, $Z = s + iy$

$\sigma(t)$ satisfying the Holder condition[3].

Let us take,

$$F(Z) = \frac{1}{2\pi i} \left[\int_0^{\infty} \frac{\bar{\sigma}(t)}{t - Z}\, dt - \int_0^{\infty} K_1(t - Z)\, e^{Z - t}\, \bar{\sigma}(t)\, dt + \int_0^{\infty} K_0(t - Z)\, e^{Z - t}\, \bar{\sigma}(t)\, dt \right] \tag{11}$$

Which is of the order of $1/|Z|$ as $|Z| \to \infty$. Throughout our analysis we shall consider that branch of $\sqrt{Z}$ which is real when arg. $Z = 0$. We ascribe argument of Z to be Zero when Z approaches the real axis from the upper side and argument to be 2π in its lower approach. The arguments of $(Z - t)$ are as follows:

$$\left.\begin{aligned} \arg(t - Z) &= -\pi \quad \text{when } Z \to s + io, \; t < s \\ &= o \quad \text{when } Z \to s + io, \; t > s \\ \arg(t - Z) &= \pi \quad \text{when } Z \to s - io, \; t < s \\ &= o \quad \text{when } Z \to s - io, \; t > s \end{aligned}\right\} \tag{12}$$

We denote by $F_t + (s)$ as the limit of the function $F(Z)$ when approaches Z the real axis in $(0, \infty)$ from upper side and $F^-(s)$, the limit of the function when Z approaches the real axis from the lower side. Now taking the limits of the function $F(Z)$ as defined in (11) and considering the limits when Z approac hes the real axis from upper and lower sides we have,

$$F + (s) = \frac{1}{2\pi i}\Big[\pi i \bar{\sigma}(s) + \int_0^\infty \frac{\bar{\sigma}(t)}{t-s}\, dt - \int_0^\infty sgn\,(s-t)\, K_1(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt$$

$$+ i\pi \int_0^s I_1(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt + \int_0^\infty K_0(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt$$

$$+ i\pi \int_0^s I_0(|s-t|)\, e^{s-t}\bar{\sigma}\, dt\Big] \tag{13a}$$

$$\bar{F}(s) = \frac{1}{2\pi i}\Big[-\pi i \bar{\sigma}(s) + \int_0^\infty \frac{\bar{\sigma}(t)}{t-s}\, dt - \int_0^\infty sgn\,(s-t)\, K_1(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt$$

$$- i\pi \int_0^s I_1(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt + \int_0^\infty K_0(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt$$

$$- i\pi \int_0^s I_0(|s-t|)\, e^{s-t}\bar{\sigma}(t)\, dt\Big] \tag{13b}$$

where K_0, K_1, I_0, I_1, have their usual meanings.

Adding and subtracting (13a) and (13b) we get the following Reiman Hilbert boundary value problem for the sectionally analytic function $F(Z)$

$$F + (s) + F^-(s) = \frac{1}{\pi i}, \quad 0 < s < \infty \tag{14}$$

and $$F+ (s) - F^-(s) = \bar{\sigma}(s) + \int_0^s \Big[[I_1(|s-t|) + I_0(|s-t|)\Big]e^{s-t}\bar{\sigma}(t)\, dt \quad 0 < s < t \tag{15}$$

It is apparent from (14) that we may take

$$F(Z) = \frac{1}{2\pi i} e^{-\epsilon Z} + \frac{A_0}{\sqrt{Z}} \tag{16}$$

where ultimately $\epsilon \to 0$ and taking $A_0 = 1$ we can find $F + (s)$ and $F^-(s)$

Substituting these value of $F + (s)$ and $F^-(s)$ from (16) to (15) we have,

$$\frac{2}{\sqrt{s}} = \bar{\sigma}(s) + \int_0^s \Big[I_1(|s-t|) + I_0(|s-t|)\Big]e^{s-t}\bar{\sigma}(t)\, dt \tag{17}$$

Taking laplace transform[4] to both sides

$$\bar{\sigma}(p) = 2\sqrt{\pi}\,/\,\sqrt{p+2} \tag{18}$$

By the inversion formula laplace transform

we get $$\bar{\sigma}(t) = 2t^{-\frac{1}{2}} e^{-2t} \tag{19}$$

Returning to original variables in terms of x and k, we have, for small values of k

$$\sigma(n) \sim 0(k^{-\frac{1}{2}} n^{-\frac{1}{2}}) \tag{20}$$

The first term of (20) is in agreement with the solution obtained by Lewis and carrier[5] for the present problem.

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