ON SOME TRANSFORMATIONS APPLIED TO THE SOLUTIONS OF THE LANE-EMDEN EQUATIONS FOR PLANE-SYMMETRIC POLYTROPES

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In this paper, we have studied a few transformations, which connect solutions of Lane-Emden equation of index n for plane-symmetric polytropic configurations in (u_θ, v_θ) —, (u_P, v_P) —, (u_ρ, y_ρ) — planes and (y_θ, z_θ) —, (y_P, z_P) — , (y_ρ, z_ρ) — planes suffixes θ , P and ρ denote the variables in which the fundamental equations of hydrostatic equilibrium have been expressed, and suffix P means we have considered our fundamental equations in terms of r & P, (u_P, v_P) and (y_P, z_P) are the variables which transform these equations into the first-order equations. The same is applicable for the variables ρ and θ.

The author[1] has recently shown that, like spherical and cylindrical polytropes, if we study the structure of a plane-symmetric polytrope in terms of r and P, then our polytropic equation of state $P = K\rho^{1+1/n}$ will be relevant at the origin of the configuration for $n \to -1$ and solutions of an equation equivalent to the Lane-Emden equation for $n \to -1$ will govern the origin. If we study the structure in terms of r and ρ the solutions for $n \to 0$ will be relevant at the origin irrespective of the index of the configuration. It has further been found that if we study the structure of polytropic configurations in (ξ, θ) — variables then there is some difficulty at the origin. In terms of r and ρ the explicit solutions governing the origin are not known. Solutions for $n \to 0$ and -1 in (u_ρ, v_ρ) — variables cannot be directly derived from solutions in (u_P, v_P) — variables. Solutions for $n \to 0$ and -1 in (r, P) — variables can be studied without any difficulty. Thus it is seen that r and P are most suitable variables for studying the structure of plane-symmetric configurations (both spherical and cylindrical also). Hence the study of a few transformations with the help of which we may immediately derive the solutions in (u_P, v_P) — and (u_ρ, v_ρ) — planes from the corresponding solutions in (u_θ, v_θ) — plane, has both mathematical and physical significance. The transformations connecting solutions in (y_P, z_P) — and (y_ρ, z_ρ) — planes with that of (y_θ, z_θ) — plane are also of importance.

First-Order Differential Equations in (u_θ, v_θ) —, (u_P, v_P) — and (u_ρ, v_ρ) — Planes for Plane-Symmetric Configurations

Lane-Emden equation of index n in (ξ_θ, θ)—variables, for plane-symmetric configurations, is given by

$$\frac{d}{d\xi_\theta}\left(\frac{d\theta}{d\xi_\theta}\right) = -\theta^n, \tag{1}$$

where the dimensionless variable ξ_θ is defined by

$$\xi_\theta = r\,\alpha_\theta^{-1}, \qquad \alpha_\theta = \left[\frac{(n+1)\,K}{4\,\pi G}\,\lambda^{(1/n)-1}\right]^{\frac{1}{2}}. \tag{2}$$

Let the two functions u_θ and v_θ, be related with the variables (ξ_θ, θ) by equations

$$u_\theta = -\frac{\xi_\theta\,\theta^n}{\theta'}, \qquad v_\theta = -\frac{\xi_\theta\,\theta'}{\theta}, \qquad \theta' = \frac{d\theta}{d\xi_\theta} \tag{3}$$

then (1) gets transformed into the first-order differential equation

$$\frac{u_\theta}{v_\theta}\,\frac{dv_\theta}{du_\theta} = -\frac{u_\theta + v_\theta + 1}{u_\theta + nv_\theta - 1}\,. \tag{4}$$

Lane-Emden equations for plane-symmetric configurations in (r, P) — and (r, ρ) — variables are

$$\frac{d}{dr}\left(P^{-\frac{n}{n+1}}\,\frac{dP}{dr}\right) = -4\,\pi GK^{-\frac{2n}{n+1}}\,P^{\frac{n}{n+1}}\,; \tag{5}$$

and

$$\frac{d}{dr}\left(\rho^{\frac{1-n}{n}}\,\frac{d\rho}{dr}\right) = -\frac{4\,\pi Gn}{K(n+1)}\,\rho \tag{6}$$

which may be re-expressed in the form

$$\frac{d}{d\xi_P}\left(P^{-\frac{n}{n+1}}\frac{dP}{d\xi_P}\right) = - P^{\frac{n}{n+1}} , \tag{7}$$

and

$$\frac{d}{d\xi_\rho}\left(\rho^{\frac{1-n}{n}}\frac{d\rho}{d\xi_\rho}\right) = - \rho. \tag{8}$$

Where the "reduced radii" ξ_P and ξ_ρ are defined by

$$r = \alpha_P\, \xi_P, \quad \alpha_P = \left[K^{\frac{2n}{(n+1)}} \Big/ 4\pi G \right]^{\frac{1}{2}} \tag{9}$$

and

$$r = \alpha_\rho\, \xi_\rho , \quad \alpha_\rho = \left[K\,(n+1) \,/\, 4\pi G n \right]^{\frac{1}{2}} \tag{10}$$

The substitutions

$$u_P = - \xi_P\, P^{\frac{2n}{n+1}} \Big/ P' , \; v_P = - \xi_P . P' / P, P' = \left(dP / d\xi_P \right) \tag{11}$$

and

$$u_\rho = - \xi_\rho\, \rho^{\frac{2n-1}{n}} \Big/ \rho' , \quad v_\rho = - \xi_\rho . \rho' / \rho , \quad \rho' = (d\rho / d\xi_\rho) , \tag{12}$$

reduce (7) and (8) into the form

$$\frac{u_P}{v_P}\,\frac{dv_P}{du_P} = - \frac{(n+1)\,u_P + v_P + (n+1)}{(n+1)\,u_P + nv_P - (n+1)} , \tag{13}$$

and

$$\frac{u_\rho}{v_\rho}\,\frac{dv_\rho}{du_\rho} = - \frac{nu_\rho + v_\rho + n}{n\,(u_\rho + v_\rho - 1)} ; \tag{14}$$

our desired first-order differential equations in (u_P, v_P) — and (u_ρ, v_ρ) — variables.

Transformations Connecting Solutions in (u_θ, v_θ) — Plane with Solutions in (u_P, v_P) — and (u_ρ, v_ρ) — Planes

From (3), (11) and (12) it is clear that there exist relations which connect the variables $\theta(u\, v_\theta)$ with the variables (u_P, v_P) and (u_ρ, v_ρ). Such relations are deduced below :

Form (2), (3), (9) and (11), we get

$$\frac{u_\theta}{u_P} = \frac{\alpha_P}{\alpha_\theta} \cdot \frac{\theta^n P'}{\theta' P^{\frac{2n}{n+1}}} \tag{15}$$

Using the relations

$$\theta' = \frac{P'}{\theta^n}\,\frac{\alpha_\theta}{\alpha_P} \cdot \frac{1}{K\,(n+1)\,\lambda^{1+\frac{1}{n}}} ; \quad P^{\frac{2n}{n+1}} = K^{\frac{2n}{n+1}}\, \lambda^2\, \theta^{2n}, \tag{16}$$

we get from (15),

$$u_\theta = u_P . \tag{17}$$

Similarly, using (2), (3), (10), (12) and the relations

$$\theta' = \frac{1}{n\lambda}\left(\frac{\alpha_\theta}{\alpha_\rho}\right) \cdot \frac{\rho'}{\theta^{n-1}} ; \quad \rho^{\frac{2n-1}{n}} = \lambda^{\frac{2n-1}{n}}\, \theta^{2n-1} \tag{18}$$

we deduce

$$u_\theta = u_\rho . \tag{19}$$

We deduce, form (16) and (18), that

$$u_P = u_\rho . \tag{20}$$

In a similar way as above, it is easy to establish the following results :

$$\left.\begin{aligned} v_\theta &= v_P \,/\, (n+1) , \\ v_\theta &= v_\rho \,/ n \\ v_P &= (n+1)\, v_\rho \,/\, n \end{aligned}\right\} \tag{21}$$

First-Order Differential Equations in (z_θ, y_θ) — (z_P, y_P) — and (z_ρ, y_ρ) — Planes for Plane-Symmetric Configurations.

The relations between the variables (z_θ, y_θ) and (ξ_θ, θ) are expressed by

$$z_\theta = \xi_\theta^{-\omega_\theta}\,\theta\,; \quad \omega_\theta = 2/(n-1)\,, \tag{22}$$

$$y_\theta = \frac{dz_\theta}{dt_\theta} = \xi_\theta^{-\omega_\theta+1} - \omega_\theta\, z_\theta\,;\ \xi_\theta = e^{t_\theta}. \tag{23}$$

Equations (22) and (23) reduce (1) to the first-order differential equations :

$$y_\theta \frac{dy_\theta}{dz_\theta} + (2\,\omega_\theta - 1)\, y_\theta + \omega_\theta\,(\omega_\theta - 1)\, z_\theta + z_\theta^{\,n}. \tag{24}$$

Let us further define

$$z_P = \xi_P^{-\omega_P}\, P\,; \quad \omega_P = 2\,(1+n)\,/\,(1-n), \tag{25}$$

$$y_P = \frac{dz_P}{dt_P} = \xi_P^{-\omega_P+1} - \omega_P\, z_P\,;\ \xi_P = e^{t_P}, \tag{26}$$

and

$$z_\rho = \xi_\rho^{-\omega_\rho}\,\rho\,; \quad \omega_\rho = 2\,n\,/\,(1-n)\,, \tag{27}$$

$$y_\rho = \frac{dz_\rho}{dt_\rho} = \xi_\rho^{-\omega_\rho+1} - \omega_\rho\, z_\rho\,; \quad \xi_\rho = e^{t_\rho}, \tag{28}$$

then we find that (7) and (8) are transformed into the first-order differential equations

$$y_P \frac{dy_P}{dz_P} + \frac{1}{2}\left(\frac{2}{\omega_P} - 1\right) z_P^{-1} y_P^{2} + \left(\frac{\omega_P - 2}{n} - 1\right) y_P + z_P \left\{\frac{\omega_P\,(\omega_P - 3 - n) + 2\,(n+1)}{2n}\right\} + z_P^{\frac{2n}{n+1}} = 0 \tag{29}$$

and

$$y_\rho \frac{dy_\rho}{dz_\rho} + \frac{2}{\omega_\rho}\, z_\rho^{-1}\, y_\rho^{2} + \left(\frac{2\omega_\rho}{n} - 1\right) y_\rho + \frac{\omega_\rho\,(\omega_\rho - n)}{n}\, z_\rho + z_\rho^{\frac{2n-1}{n}} = 0 \tag{30}$$

Transformations Connecting Solutions in (z_θ, y_θ) — Plane with Solutions in (z_P, y_P) — and (z_ρ, y_ρ) — Planes

From (22) and (25), we obtain

$$\frac{z_P}{z_\theta} = K\lambda^{1+\frac{1}{n}} \left(\frac{\alpha_\theta}{\alpha_P}\right)^{-\frac{2(1+n)}{1-n}} \xi_\theta^{-\frac{2n}{1-n}}\, \theta^{n} = K\lambda^{1+\frac{1}{n}} \left(\frac{\alpha_\theta}{\alpha_P}\right)^{-\frac{2(1+n)}{1-n}} z_\theta^{\,n},$$

that is,

$$z_P = (n+1)^{\frac{1+n}{n-1}}\, z_\theta^{\frac{n+1}{n}} \tag{31}$$

In a similar way we obtain, from first equation in (21) and (27),

$$z_\rho = n^{\frac{n}{n-1}}\, z_\theta^{\,n}. \tag{32}$$

Hence we deduce, from (31) and (32), that

$$z_P = \left(\frac{n+1}{n}\right)^{\frac{n+1}{n-1}} \quad z_\rho^{\,1+\frac{1}{n}} . \tag{33}$$

Further, we can express the variables y_p and y_θ in terms of (z_P, ξ_P) and (z_θ, ξ_θ) respectively in the form

$$y_P = \xi_P \frac{dz_P}{d\xi_P} \tag{34}$$

and

$$y_\theta = \xi_\theta \frac{dz_\theta}{d\xi_\theta} . \tag{35}$$

But since

$$\xi_P = \frac{\alpha_\theta}{\alpha_P} \xi_\theta; \; z_P = (n+1)^{\frac{n+1}{n-1}} z_\theta^{\,n+1} . \tag{36}$$

hence, from (34) and (35) we obtain

$$y_P = (n+1)^{\frac{2n}{n-1}} \; y_\theta \, z_\theta^{\,n} . \tag{37}$$

Similarly, we get

$$y_\rho = n^{\frac{2n-1}{n-1}} \; y_\theta \, z_\theta^{\,n-1}, \qquad y_P = \left[\frac{n+1}{n}\right]^{\frac{2n}{n-1}} y_\rho \, z_\rho^{\,1/n} . \tag{38}$$

Thus with the help of the transformations given in (31), (32), (33), (37) and (38) solutions of the Lane-Emden equations for plane-symmetric configurations in (z_P, y_P) — and (z_ρ, y_ρ) — variables are deduceable from the corresponding solutions in (z_θ, y_θ)—variables.

CONCLUSIONS

(*i*) Substitutions given in (3), (22) and (23) reduce the Lane-Emden equation of index n in (ξ_θ, θ) —variables [eqn. (1)] for plane-symmetric configurations into the first-order differential equations (4) and (24) respectively.

(*ii*) Substitutions given in (11), (25) and (26) reduce the Lane-Emden equation of index n in (ξ_P, P) — variables [eqn. (7)] into the first-order differential equations (13) and (29) respectively.

(*iii*) Substitutions given in (12), (27) and (28) reduce the Lane-Emden equation of index n in $(\xi_\rho, \rho,)$ — — variables [eqn. (8)] into the first-order differential equations (14) and (30) respectively.

(*iv*) Equations (17), (19), (20) and (21) enable us to derive the solutions of the Lane-Emden equations in (u_P, v_P) — and (u_ρ, v_ρ) — planes from the corresponding solutions in (u_θ, v_θ) — plane for the plane-symmetric configurations.

(*v*) Equations (31), (32), (33), (37) and the set of equations in (38) connect the solutions in (z_P, y_P) —, (z_ρ, y_ρ) — and (z_θ, y_θ) — planes.

(*vi*) Transformations given in (17), (19), (20), (21), (31), (32), (33), (37) and (38) are also applicable for spherical polytropes[2].

As has already been concluded above in brief the main idea underlying our investigation is : If we know, for a given n, the solutions of plane-symmetric polytropes (such as Saturn ring system and Laplacian disc cosmogonies) in (u_θ, v_θ) — plane, then the corresponding solutions in (u_P, v_P) — and (u_ρ, v_ρ)— planes can be derived directly with the help of the relations (17), (19), (20) and (21). In a similar way, having the knowledge of the solutions in $(z\theta, y\theta)$ — plane, the solutions in (z_P, y_P) —and (z_ρ, y_ρ) — planes can be easily obtained with the help of equations (31), (32), (33), (37) and (38). In other words the mass, the radius, distributions of pressure and density, and temperature, etc., which describe the structure of plane-symmetric astrophysical bodies (as mentioned above) can be studied easily in different planes in view of the set of equations (17), (19), (20), (21), (31), (32), (33), (37) and (38).

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