ON DIFFERENTIAL EFFECTS OF THE SHOCK WAVE IN CONDUCTING FLUID

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The differential effects of the shock wave in conducting fluid have been discussed using purely a vector technique. While discussing the vorticity and current density generated behind the shock wave, it has been observed that expression for the vorticity depends on the thermodynamical behaviour of the fluid.

Some authors[1-5] have discussed the differential effects of shock waves using tensorial technique in which vector quantities-vorticity and current density are obtained by the contraction of indices of tensor quantities of order two. In the present paper vector technique method proposed by Hayes[6] and developed by the authors[7-8] has been used. The motive behind it is that the vector quantities can be obtained directly from the set of vector equations and the determination of the tensor quantities of higher order are not necessary. This very technique is used for the determination of the derivatives of the flow and field parameters vorticity, current density and the curvatures of stream line and magnetic line.

DISCUSSION ON THE FLOW

The equations governing the flow of an ideal conducting fluid are[9]

$$\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \bar{v}) = 0 \tag{1}$$

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho (\bar{v} \cdot \nabla) \bar{v} + \nabla p + \frac{1}{8\pi} \nabla H^2 - \frac{1}{4\pi} (\bar{H} \cdot \nabla) \bar{H} = 0, \tag{2}$$

$$\frac{\partial \bar{H}}{\partial t} + (\bar{v} \cdot \nabla) \bar{H} + \bar{H} (\nabla \cdot \bar{v}) - (\bar{H} \cdot \nabla) \bar{v} = 0, \tag{3}$$

$$\frac{\partial \eta}{\partial t} + (\bar{v} \cdot \nabla) \eta = 0, \tag{4}$$

where p, ρ, $\bar{v}$, $\bar{H}$ and η stand for the vorticity pressure, density, velocity vector, magnetic field vector and entropy respectively of the fluid under consideration.

From the thermodynamical relation of the fluid we know that

$$\eta = \eta (p, \rho). \tag{5}$$

In view of the relation (5) the equation of state i.e., (4) reduces to

$$\frac{\partial p}{\partial t} + (\bar{v} \cdot \nabla) p = C^2 \left\{ \frac{\partial \rho}{\partial t} + (\bar{v} \cdot \nabla) \rho \right\}, \tag{6}$$

where C^2 stands for $\left(- \frac{\partial \eta}{\partial p} \right)_{\rho = \text{Constant}} \Big/ \left(\frac{\partial \eta}{\partial \rho} \right)_{\rho = \text{Constant}}$

If the velocity of the shock be $\hat{G}\hat{n}$ ($\hat{n}$ is unit normal vector to shock surface), then the material derivative of any flow or field parameter along the normal trajectory to the shock surface is given by[10]

$$\frac{\delta f}{\delta t} = G \frac{\partial f}{\partial n} + \frac{\partial f}{\partial t}, \tag{7}$$

where f is any flow or field parameter including the components of the velocity and magnetic field vectors.

Resolving the del operator along normal to the shock surface and in the plane of the shock surface we have[6]

$$\nabla \equiv \hat{n} \frac{\partial}{\partial n} + \nabla_t, \tag{8}$$

where ∇_t denotes the resolved part of ∇ in the tangent plane to the shock surface.

In view of the relations (7) and (8), the equations (1) to (4) reduce to

$$V_n \frac{\partial \rho}{\partial n} + \rho \hat{n} . \frac{\partial \bar{v}}{\partial n} + \alpha = 0, \tag{9}$$

$$\rho V_n \frac{\partial \bar{v}}{\partial n} + \hat{n} \frac{\partial p}{\partial n} + \frac{1}{4\pi} \left\{ \hat{n} \left(\bar{H} . \frac{\partial \bar{H}}{\partial n} \right) - H_n \frac{\partial \bar{H}}{\partial n} \right\} + \bar{A} = 0, \tag{10}$$

$$V_n \frac{\partial \bar{H}}{\partial n} + \bar{H} \left(\hat{n} . \frac{\partial \bar{v}}{\partial n} \right) - H_n \frac{\partial \bar{v}}{\partial n} + \bar{B} = 0, \tag{11}$$

$$V_n \frac{\partial p}{\partial n} - C^2 V_n \frac{\partial \rho}{\partial n} + \beta = 0, \tag{12}$$

where, V_n and H_n stands for the normal components of velocity and magnetic field with respect to shock.

$$\left.\begin{aligned} \alpha &= \frac{\delta \rho}{\delta t} + (\bar{v} . \nabla_t) \rho + \rho (\nabla_t . \bar{v}), \quad \beta = \frac{\delta p}{\delta t} - C^2 \frac{\delta \rho}{\delta t} + (\bar{v} . \nabla_t) p - C^2 (\bar{v} . \nabla_t) \rho, \\ \bar{A} &= \rho \frac{\delta \bar{v}}{\delta t} + \rho (\bar{v} . \nabla_t) \bar{v} + \nabla_t p + \frac{1}{4\pi} \left\{ \tfrac{1}{2} \nabla_t H^2 - (\bar{H} . \nabla_t) \bar{H} \right\}, \\ \bar{B} &= \frac{\delta \bar{H}}{\delta t} + (\bar{v} . \nabla_t) \bar{H} + \bar{H} (\nabla_t . \bar{v}) - (\bar{H} . \nabla_t) \bar{v}. \end{aligned}\right\} \tag{13}$$

The resolved components of (10) and (11) along the normal to the shock surface are

$$\rho V_n \hat{n} . \frac{\partial \bar{v}}{\partial n} + \frac{\partial p}{\partial n} + \frac{1}{4\pi} \left(\bar{H} . \frac{\partial \bar{H}}{\partial n} - H_n \hat{n} . \frac{\partial \bar{H}}{\partial n} \right) + \bar{A} . \hat{n} = 0, \tag{14}$$

$$V_n \hat{n} . \frac{\partial \bar{H}}{\partial n} + \bar{B} . \hat{n} = 0. \tag{15}$$

Eliminating $\hat{n} . \frac{\partial \bar{H}}{\partial n}$ from (14) and (15) we get

$$\rho V_n \hat{n} . \frac{\partial \bar{v}}{\partial n} + \frac{\partial p}{\partial n} + \frac{1}{4\pi} \left(\bar{H} . \frac{\partial \bar{H}}{\partial n} + \frac{H_n}{V_n} \bar{B} . \hat{n} \right) + \bar{A} . \hat{n} = 0. \tag{16}$$

The resolved components of (10) and (11) along the magnetic line are :

$$\rho V_n \bar{H} . \frac{\partial \bar{v}}{\partial n} + H_n \frac{\partial p}{\partial n} + \bar{A} . \bar{H} = 0, \tag{17}$$

and

$$V_n \bar{H} . \frac{\partial \bar{H}}{\partial n} + H^2 \hat{n} . \frac{\partial \bar{v}}{\partial n} - H_n \bar{H} . \frac{\partial \bar{v}}{\partial n} + \bar{H} . \bar{B} = 0. \tag{18}$$

Eliminating $\bar{H} . \frac{\partial \bar{v}}{\partial n}$ from (17) and (18) we get,

$$V_n \bar{H} . \frac{\partial \bar{H}}{\partial n} + H^2 \hat{n} . \frac{\partial \bar{v}}{\partial n} + \frac{H_n}{\rho V_n} \left(H_n \frac{\partial p}{\partial n} + \bar{H} . \bar{A} \right) + \bar{H} . \bar{B} = 0. \tag{19}$$

Solving (16) and (19), we have

$$\begin{aligned} \hat{n} . \frac{\partial \bar{v}}{\partial n} \left\{ H^2 - 4 \pi \rho V_n^2 \right\} + \frac{\partial p}{\partial n} \left\{ \frac{H_n^2}{\rho V_n} - 4 \pi V_n \right\} + \bar{H} . \bar{B} - H_n \bar{B} . \hat{n} + \\ + \frac{H_n}{\rho V_n} \bar{H} . \bar{A} - 4 \pi V_n \bar{A} . \hat{n} = 0. \end{aligned} \tag{20}$$

But in view of the equation (9), the equation (12) gives

$$\frac{V_n}{C^2}\frac{\partial p}{\partial n}+\rho\hat{n}\cdot\frac{\partial \bar{v}}{\partial n}+\alpha+\frac{1}{C^2}\beta=0, \tag{21}$$

$$4\pi\rho C^2\left\{M_n^2\left(b^2-1\right)-\left(b^2_n-1\right)\right\}\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}-\frac{1}{V_n}\left(C^2\alpha+\beta\right)+$$
$$+\bar{H}\cdot\left\{\frac{H_n}{\rho V_n}\bar{A}+\bar{B}\right\}-4\pi V_n\,\hat{n}\cdot\left(\bar{A}+\frac{H_n}{4\pi V_n}\bar{B}\right)=0, \tag{22}$$

where $$M_n^2=\frac{V_n^2}{C^2}\text{ and }b^2=\frac{H^2}{4\pi\rho V^2_n}$$

DETERMINATION OF THE DIFFERENTIAL EFFECTS OF THE SHOCK WAVE

It is well-known that the appearance of the shock wave is associated with the change in the vorticity, current density, derivatives of the flow and field parameters and the shape of the path of the fluid particles. For the differential effects of the shock wave it is sufficient to determine the jumps in the vorticity and current density, the normal derivatives of the flow and field parameters the curvatures and torsions of the stream and magnetic lines. In this connection we have the following theorems :

Theorem 1

The normal derivatives of the flow and field parameters are given by

$$\frac{\partial\rho}{\partial n}=-\frac{1}{V_n}\left\{\rho\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}+\alpha\right\}, \tag{23}$$

$$\frac{\partial p}{\partial n}=-\frac{C^2}{V_n}\left\{\rho\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}+\alpha+\frac{\beta}{C^2}\right\}, \tag{24}$$

$$\frac{\partial\bar{H}}{\partial n}=\bar{C}+D\hat{n}, \tag{25}$$

$$\frac{\partial\bar{v}}{\partial n}=\bar{E}+F\hat{n}, \tag{26}$$

where

$$\bar{C}=-\frac{1}{V_n(1-b^2_n)}\left\{\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}\bar{H}_t+\bar{B}+\frac{H_n}{\rho V_n}\bar{A}\right\},$$

$$D=\frac{H_n}{V_n}\left\{\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}\frac{M^2_n+1-b^2_n}{M^2_n}+\frac{\bar{H}}{4\pi\rho V^2_n}\cdot\left(\bar{B}+\frac{H_n}{\rho}\bar{A}\right)+\frac{1}{\rho M^2_n}\left(\alpha+\frac{\beta}{C^2}\right)\right\},$$

$$\bar{E}=\frac{1}{1-b^2_n}\left\{\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}\frac{b^2_n\bar{H}_t}{H_n}+\frac{b^2_n}{H_n}\bar{B}+\frac{\bar{A}}{\rho V_n}\right\},\quad F=\frac{V_n}{H_n}D-\hat{n}\frac{\partial\bar{v}}{\partial n},$$

and $\hat{n}\,\frac{\partial\bar{v}}{\partial n}$ is given by the equation (22).

Proof :—Equations (23) and (24) can be obtained easily with the help of (9) and (12). Substituting the value of $\frac{\partial p}{\partial n}$ from (24) and $\bar{H}\,\frac{\partial\bar{H}}{\partial n}$ from (19) in (10) and (11) respectively and solving them for $\frac{\partial\bar{v}}{\partial n}$ and $\frac{\partial\bar{H}}{\partial n}$ we get the relations (25) and (26).

Theorem 2

The expressions for the vorticity and current density generated behind the shock wave are given by

$$\bar{\omega}=\frac{1}{(1-b^2_n)}\left\{\left(\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}\right)\frac{b^2_n\bar{H}_t}{H_n}+\frac{b^2_n\bar{B}}{H_n}+\frac{\bar{A}}{\rho V_n}\right\}\times\hat{n}+\nabla_t\times\bar{v}, \tag{27}$$

$$\bar{J}=\frac{1}{4\pi V_n(1-b^2_n)}\left\{\left(\hat{n}\cdot\frac{\partial\bar{v}}{\partial n}\right)\bar{H}_t+\bar{B}+\frac{H_n}{\rho V_n}\bar{A}\right\}\times\hat{n}+\nabla_t\times\bar{H}. \tag{28}$$

Proof :—We know from the definitions of the vorticity and current density vectors that[9]

$$\bar{\omega}=\nabla\times\bar{v},\quad(29);\qquad \bar{J}=\frac{1}{4\pi}\nabla\times\bar{H}\cdot \tag{30}$$

In view of relation (8), the equations (29) and (30) yield

$$\bar{\omega}=\hat{n}\times\frac{\partial\bar{v}}{\partial n}+\nabla_t\times\bar{v}, \tag{31}$$

$$4\pi \bar{J} = \hat{n} \times \frac{\partial \bar{H}}{\partial n} + \nabla_t \times \bar{H} . \qquad (32)$$

Substituting the values from (25) and (26) in (31) and (32) and remembering that $\hat{n} \times \hat{n} = 0$, we get the required equations.

Theorem 3

The jumps in vorticity and current density across a magnetogasdynamic shock wave depend on the thermodynamical behaviour of the fluid. (This theorem is in contrast to the similar theorem obtained in non-conducting fluid by Truesdell[11] and the authors[8].)

Proof:—The expressions for the vorticity and current density calculated above include $\hat{n} \cdot \frac{\partial \bar{v}}{\partial n}$. But for the determination of $\hat{n} \cdot \frac{\partial \bar{v}}{\partial n}$ we have to use energy equation.

Theorem 4

The curvatures of the stream line and magnetic line behind the magnetogasdynamic shock wave are given by

$$K^2 = \frac{1}{v^2}\left[\left\{ V_n \bar{E} + (\bar{v} \cdot \nabla_t)\,\bar{v} \right\}^2 + V^2_n F^2 + 2\, V_n F\hat{n} \cdot \left\{ V_n \bar{E} + (\bar{v} \cdot \nabla_t)\,\bar{v} \right\}\right], \qquad (33)$$

$$K'^2 = \frac{1}{H^2}\left[\left\{ H_n \bar{C} + (\bar{H} \cdot \nabla)\,\bar{H} \right\}^2 + H^2_n D^2 + 2\, H_n D\hat{n} \cdot \left\{ H_n \bar{C} + (\bar{H} \cdot \nabla)\,\bar{H} \right\}\right], \qquad (34)$$

where $\qquad v = |\,\bar{v}\,|$ and $H = |\,\bar{H}\,|$

Proof:—From the definition of the curvature of a curve we have

$$K^2 = \frac{\partial \bar{v}}{\partial s} \cdot \frac{\partial \bar{v}}{\partial s}, \quad (35); \qquad K'^2 = \frac{\partial \bar{H}}{\partial s'} \cdot \frac{\partial \bar{H}}{\partial s'} . \qquad (36)$$

where, s and s' denote the arc length along stream line and magnetic line respectively. But in view of relation (8) the derivatives along stream line and magnetic line are

$$\frac{\partial \bar{v}}{\partial s} = \frac{1}{v}\left\{ V_n \frac{\partial \bar{v}}{\partial n} + (\bar{v} \cdot \nabla_t)\,\bar{v} \right\}, \quad (37); \qquad \frac{\partial \bar{H}}{\partial s'} = \frac{1}{H}\left\{ H_n \frac{\partial \bar{H}}{\partial n} + (\bar{H} \cdot \nabla_t)\,\bar{H} \right\}, \qquad (38)$$

Substituting the values of $\frac{\partial \bar{v}}{\partial n}$ and $\frac{\partial \bar{H}}{\partial n}$ from relations (25) and (26) in above equations we get

$$\frac{\partial \bar{v}}{\partial s} = \frac{1}{v}\left\{ V_n \bar{E} + (\bar{v} \cdot \nabla_t)\,\bar{v} + V_n F\hat{n} \right\}, \qquad (39)$$

$$\frac{\partial \bar{H}}{\partial s'} = \frac{1}{H}\left\{ H_n \bar{C} + (\bar{H} \cdot \nabla_t)\,\bar{H} + H_n D\hat{n} \right\}. \qquad (40)$$

In view of equations (39) and (40) we get the required expressions for K and K' as given in equations (33) and (34).

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