DESIGN OF SPRINGS BY OPTIMIZATION

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Springs are extensively used in various defence equipments ranging from tanks to delicate instruments. This necessitates their optimum design with respect to weight or volume. The present paper discusses the design of conical helical springs leaf springs and torsional helical springs using optimization procedure.

Springs of different types occupy an important place in mechanical devices and machines. Depending on the application, the springs may be of the helical, leaf, torsion, etc. types. The selection of springs is further restricted by various considerations such as weight, space, length, etc. For example, a clutch spring is required to have maximum deflection for least space whereas the springs used in vehicles should have minimum possible weight to perform the required task. Conical compression springs are most useful for applications where a varying load deflection rate is needed. Such conical springs have smaller solid height, greater lateral stability, reduce sensitivity to buckling, and constantly increasing natural period of vibrations as each coil bottoms. Such types of springs can be used safely in automobiles as isolating devices.

In this paper designs of different springs are presented by optimization with respect to parameters like weight and volume.

The optimization procedure for conical helical springs given in this paper follows that given by Hinkle and Morse[1] and Rao[2]. It is shown that the curves obtained have a similar trend as those of Rao[2].

NOTATIONS

A = design constant for optimum weight and volume

B = constant

C = spring index = D/d

d = wire diameter

D_1 = smallest mean diameter of the coil

D_2 = greatest mean diameter of the coil

D_m = mean coil diameter

G = torsional modulus of elasticity

H = Height of the cone

i = number of inactive coils

K_1, K = Wahl's correction factor

k_s = direct shear correction factor

m = material constant

F, P = maximum spring load

W = weight of the spring

ϕ = material constant which depends on the serverity of the loading

ρ = density of spring wire material

τ_{max} = maximum shear stress in the spring

τ = allowable shear stress

δ = deflection under maximum load

M = Bending moment in kg. cm.

a = distance from the load line to the spring axis

E = modulus of elasticity

L = length of the leaf spring

b = width of the leaf spring

n = number of active coils

V_s = volume occupied by the spring when compressed solid

t = thickness

n_g = graduated leaves

n_f = fully graduated leaves

DESIGN OF CONICAL HELICAL SPRINGS

Minimum Weight

The expression for weight of a conical helical spring is given by (see Appendix)

$$W = \rho . \frac{\pi D_2}{2H} (H + h) (n + i) \frac{\pi}{4} d^2$$

or

$$\frac{8W}{\rho \pi^2} \frac{H}{(H+h)} = D_2 (n + i) d^2 \quad (1)$$

where $D_2 = Cd$. putting this value in (1)

$$\frac{8W}{\rho \pi^2} \frac{H}{(H+h)} = Cnd^3 + Cid^3 \quad (2)$$

The maximum shear stress in the spring is given by the following equation

$$\tau_{max} = \frac{8KPC}{\pi d^2} \quad (3)$$

where

$$K = \frac{4C - 1}{4C - 4} + \frac{0{\cdot}615}{C} \quad (4)$$

Wahl's correction factor K, used in (4) is normally used for cylindrical springs. But the same can be used for these conical springs also where the cross-section dimensions are small in comparison to the mean coil diameter[5].

For design purpose

$$\tau_{max} < \tau \quad (5)$$

The shear stress τ decreases with an increase in diameter which is expressed as

$$\tau = \frac{\phi}{d^m} \quad (6)$$

From (3), (5) and (6)

$$d = \left(\frac{8KPC}{\pi \phi} \right)^{\frac{1}{2-m}} \quad (7)$$

The deflection, δ, for conical spring is taken as

$$\delta = \frac{2nC^3 P}{d\,GK}, \quad n = \frac{d\,GK\,\delta}{2C^3 P} \quad (8)$$

In the case of a conical spring when the deflection is to be accurately calculated then it is necessary to substitute $(D_1^2 + D_2^2)$ for D^2 and $\pi n (D_1 + D_2)$ for $\pi n D$, D_1 and D_2 being the smallest and greatest mean diameters of the coils. But for the case where the cross-section dimension of the spring wire is small in comparison to the mean coil diameter then the above formula gives fairly accurate results[5].

Substituting the value of d and n from (7) and (8) in (2), we get

$$\frac{8W}{\rho\pi^2}\frac{H}{(H+h)}=\frac{G\delta}{2P}\frac{K}{C^2}\left(\frac{8P}{\pi\phi}\right)^{\frac{4}{2-m}}K^{\frac{4}{2-m}}C^{\frac{4}{2-m}}+Ci\left(\frac{8P}{\pi\phi}\right)^{\frac{3}{2-m}}\cdot K^{\frac{3}{2-m}}C^{\frac{3}{2-m}}$$

This equation can be written as:

$$\frac{8W}{\rho\pi^2}\frac{H}{(H+h)i}\left(\frac{\pi\phi}{8P}\right)^{\frac{3}{2-m}}=\frac{G\delta}{2Pi}\left(\frac{8P}{\pi\phi}\right)^{\frac{1}{2-m}}K^{\frac{6-m}{2-m}}C^{\frac{2m}{2-m}}+K^{\frac{3}{2-m}}C^{\frac{5-m}{2-m}}$$

or

$$BW=AC^{\frac{2m}{2-m}}K^{\frac{6-m}{2-m}}+C^{\frac{5-m}{2-m}}K^{\frac{3}{2-m}} \tag{9}$$

$$A=\frac{G\delta}{2Pi}\left(\frac{8P}{\pi\phi}\right)^{\frac{1}{2-m}};\ B=\frac{8}{\rho\pi^2(1+h/H)i}\left(\frac{\pi\phi}{8P}\right)^{\frac{3}{2-m}} \tag{10}$$

To optimize for weight, (9) is differentiated with respect to C, and by putting $\frac{dW}{dC}=0$, we get

$$A=-\frac{C^{\frac{5-3m}{2-m}}}{K^{\frac{3-m}{2-m}}}\left[\frac{3\frac{dK}{dC}+\frac{K}{C}(5-m)}{\frac{dK}{dC}(6-m)+2m\frac{K}{C}}\right] \tag{11}$$

The value of $\frac{dK}{dC}$ is obtained by differentiating (4) with respect to C

$$\frac{dK}{dC}=\frac{-1{\cdot}365C^2+1{\cdot}23C-0{\cdot}615}{C^2(C-1)^2} \tag{12}$$

For light services the effect of curvature is neglected and only direct shear stress is considered. (7) can be written in this form

$$d=\left(\frac{8K_S PC}{\pi\phi}\right)^{\frac{1}{2-m}} \tag{13}$$

where

$$K_s=1+\frac{0{\cdot}5}{C} \tag{14}$$

To obtain the optimum value for light condition, K is replaced by Ks in (11). Therefore the value of A is given by

$$A=-\frac{C^{\frac{5-3m}{2-m}}}{K_s^{\frac{3-m}{2-m}}}\left[\frac{3\frac{dK_s}{dC}+\frac{K_s}{C}(5-m)}{\frac{dK_s}{dC}(6-m)+2m\frac{K_s}{C}}\right] \tag{15}$$

From (14)

$$\frac{dK_s}{dC}=-\frac{0{\cdot}5}{C^2} \tag{16}$$

Minimum Volume for Conical Helical Spring

The expression for volume is given by the following equation (4)

$$\frac{12V_s}{\pi h}=nd^2(C^2+C+1)+id^2(C^2+C+1) \tag{17}$$

By putting the value of d and n from (7) and (8) in (17), we get

$$B_1 V_s = A K^{\frac{3}{2-m}} C^{\frac{1+m}{2-m}} + A K^{\frac{3}{2-m}} C^{\frac{2m-1}{2-m}} + A K^{\frac{3}{2-m}} C^{\frac{3m-3}{2-m}} +$$

$$+ K^{\frac{2}{2-m}} C^{\frac{6-2m}{2-m}} + K^{\frac{2}{2-m}} C^{\frac{4-m}{2-m}} + K^{\frac{2}{2-m}} C^{\frac{2}{2-m}} \tag{18}$$

where $A = \dfrac{G\,\delta}{2\,P\,i} \left(\dfrac{8\,P}{\pi\,\phi} \right)^{\frac{1}{2-m}}$ $\quad B_1 = \dfrac{12}{\pi h} \left(\dfrac{\pi\,\phi}{8\,\pi} \right)^{\frac{2}{2-m}}$

Differentiating (18) with respect to C and putting $\dfrac{d\,V_s}{d\,C} = 0$, we get

$$A = - \frac{C^{\frac{5-3m}{2-m}}}{K^{\frac{1}{2-m}}} \left\{ \frac{\dfrac{dK}{dC}\left[2 + 2\dfrac{K}{C} + 2\dfrac{K}{C^2}\right] + \dfrac{K}{C}\left[(6-2m) + \left(\dfrac{4-m}{C}\right) + \dfrac{2}{C^2}\right]}{\dfrac{dK}{dC}\left[1 + \dfrac{3K}{C} + \dfrac{3K}{C^2}\right] + \dfrac{K}{C}\left[(1+m) + \left(\dfrac{2m-1}{C}\right) + \left(\dfrac{3m-3}{C^2}\right)\right]} \right\} \tag{19}$$

For light services replace K by K_s in (19).

DESIGN FOR TORSION SPRING

The weight expression for this case is given by

$$W = \rho\,(n + i)\,\frac{\pi}{4}\,d^2\,\pi\,D_m \tag{20}$$

The maximum stress in a spring is given by the following equation[5]

$$\tau_{max} = K_1 \frac{32\,M}{\pi\,d^3} \tag{21}$$

and deflection is given by

$$\delta = \frac{64\,M\,D_m\,a\,n}{E\,d^4} \tag{22}$$

The maximum stress is taken to be in the following form

$$\tau = \frac{\phi}{d^m} \qquad d = \left(K_1 \frac{32\,M}{\pi\,\phi} \right)^{\frac{1}{3-m}} \tag{23}$$

Substituting the values of d and n from (23) and (22) in (20), we get

$$\frac{4\,W}{\rho\,\pi^2} = \frac{E\,\delta}{64\,Ma} \left\{ K_1 \frac{32\,M}{\pi\,\phi} \right\}^{\frac{6}{3-m}} + i \left\{ K_1 \frac{32\,M}{\pi\,\phi} \right\}^{\frac{3}{3-m}} C$$

$$B\,W = A\,K_1^{\frac{6}{3-m}} + K_1^{\frac{3}{3-m}}\,C \tag{24}$$

where $A = \dfrac{E\,\delta}{64\,Ma}\,\dfrac{1}{i} \left\{ \dfrac{32\,M}{\pi\,\phi} \right\}^{\frac{3}{3-m}}$

$B = \dfrac{4}{\rho\,\pi^2}\,\dfrac{1}{i} \left\{ \dfrac{\pi\,\phi}{32\,M} \right\}^{\frac{3}{3-m}}$

By differentiating (24) with respect to C and putting $\frac{dW}{dC} = 0$, we get

$$A = -\frac{K_1^{\frac{3}{3-m}} + \frac{3}{3-m} K_1^{\frac{m}{3-m}} \cdot C \cdot \frac{dK_1}{dC}}{\frac{6}{3-m} K_1^{\frac{3+m}{3-m}} \, dK_1/dC} \qquad (25)$$

when $$m = 0{\cdot}22,\ A = -\frac{K_1^{1\cdot08} + K_1^{\cdot0792} \cdot dK_1/dC \cdot C \times 1{\cdot}08}{2{\cdot}16\ K_1^{1\cdot16}\ dK_1/dC}$$

where K_1 is given by the following equation

$$K_1 = \frac{4C^2 - C - 1}{4C(C-1)}$$

$$\frac{dK_1}{dC} = \frac{-3C^2 + 2C - 1}{4C^2(C-1)^2} \qquad (26)$$

To set the optimum weight of the spring put dK_1/dC from (26) in (25), and we get the required result.

The value of A against C is shown plotted in Fig. 1 for $m = 0{\cdot}22$.

DESIGN FOR LEAF SPRING

The equation for maximum stress and deflection for leaf spring is given by[5]

$$\tau_{max} = \frac{18\,FL}{bt^2\,(2\,n_g + 3\,n_f)} \qquad (27)$$

$$\delta = \frac{12\,FL^3}{bt^3\,E\,(2\,n_g + 3\,n_f)} \qquad (28)$$

The weight expression is given by

$$W = \rho\ m_1\ m_2\ b^3\,(2\,n_g + 3\,n_f) \qquad (29)$$

where

$$L = m_1\,b \qquad (30)$$

$$t = m_2\,b \qquad (31)$$

Putting the value of $(2\,n_g + 3n_f)$ from (28) in (29), we get

$$\frac{E\,m_2^2\,W}{\rho\,b^2\,12\,F} = \frac{m_1^4}{\delta} \qquad (32)$$

Differentiating (32) with respect to m_1 and keeping m_2 constant

$$\frac{dW}{dm_1} = \frac{1}{\delta}\,4\,m_1^3 - \frac{m_1^4}{\delta^2}\left(\frac{d\delta}{dm_1}\right) \qquad (33)$$

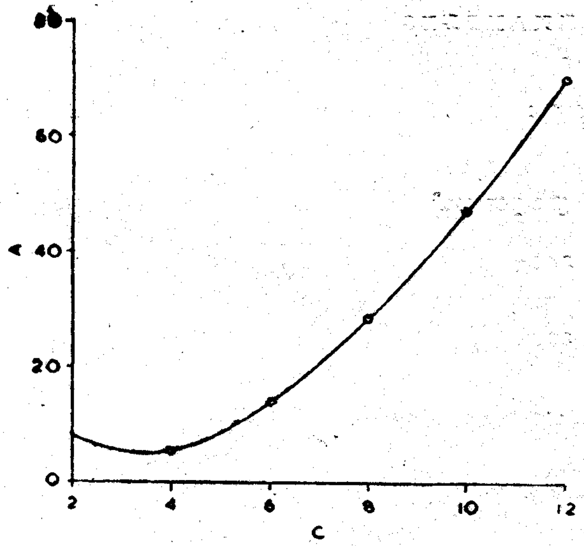

FIG. 1—Spring index for minimum weight.

The value of $\frac{d\delta}{dm_1}$ is obtained by differentiating (28) with respect m_1

$$\frac{d\delta}{dm_1} = \frac{36\,F}{m_2^3\,E}\left(\frac{m_1^2}{2\,n_g + 3\,n_f}\right) \qquad (34)$$

Putting $\frac{dW}{dm_1} = 0$ in (33), we get

$$\frac{1}{\delta} 4 m_1^3 - \frac{m_1^4}{\delta^2} \left\{ \frac{36 F}{m_2^3 E} \frac{m_1^2}{(2 n_g + 3 n_f)} \right\} = 0$$

$$\frac{4 m_1^3}{\delta} \left\{ 1 - \frac{9 F m_1^3}{\delta m_2^3 E (2 n_g + 3 n_f)} \right\} = 0$$

$$\left(\frac{m_2}{m_1} \right)^3 = \frac{9 F}{\delta E (2 n_g + 3 n_f)} \qquad (35)$$

$$t = L \left\{ \frac{9 F}{\delta E (2 n_g + 3 n_f)} \right\}^{1/3}$$

STEPS FOR THE DESIGN OF A SPRING BY OPTIMIZATION PROCEDURE

From (35) it can be seen that for optimum design of the leaf spring, it is necessary to specify the maximum permissible deflection, δ. Then with the help of Fig. 2, the value of m_1/m_2 can be calculated and hence the value of t.

To use the present optimization procedure in an actual design of a spring, the following steps are taken :

(1) Select a suitable spring index.

(2) Find out the corresponding value of A from the graph.

(3) Use the values of K, P, C and ϕ in (7) to get the value of d.

(4) D_2 is calculated from C and d.

(5) With the given values of D_2 and D_1 the design is thus completed.

NUMERICAL EXAMPLES

Example I

Design a conical spring for the following conditions :

(Data taken from ref. 2)

Load $P = 40$ Kgf, Deflection $\delta = 15$ mm, $\rho = 7 \cdot 8$ gm per cm^3

Material—hard spring steel $G = 8100$ Kgf per mm^2

Ends squared, Service condition—severe.

For severe services

$$d = \left(\frac{8 KPC}{\pi \phi} \right)^{\frac{1}{2-m}}$$

here
$$m = 0 \cdot 22, \ \frac{1}{2-m} = 0 \cdot 562$$

$$\phi = 0 \cdot 8 \cdot 50 = 40.$$

$$K = \frac{4C-1}{4C-4} + \frac{0 \cdot 615}{C} = 1 \cdot 253 \text{ for } C = 6.$$

Therefore
$$d = \left(\frac{8 \times 1 \cdot 253 \times 40 \times 6}{\pi \times 40} \right)^{\cdot 552}$$

$$= 5 \cdot 28 \text{ mm.}$$

Using (8), number of coils.

$$n = \frac{d G K \delta}{2 C^3 P}$$

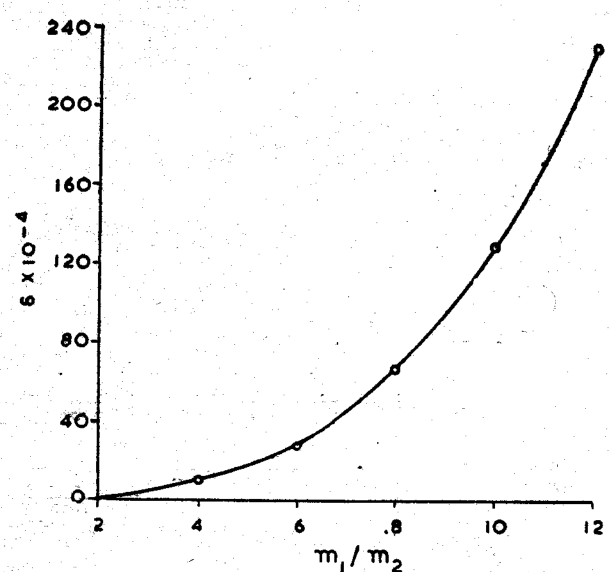

FIG. 2—Mass ratio for minimum deflection.

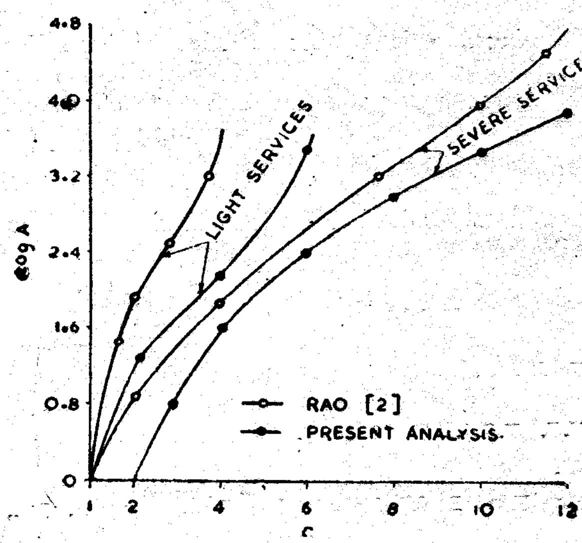

FIG. 3—Spring index for minimum weight.

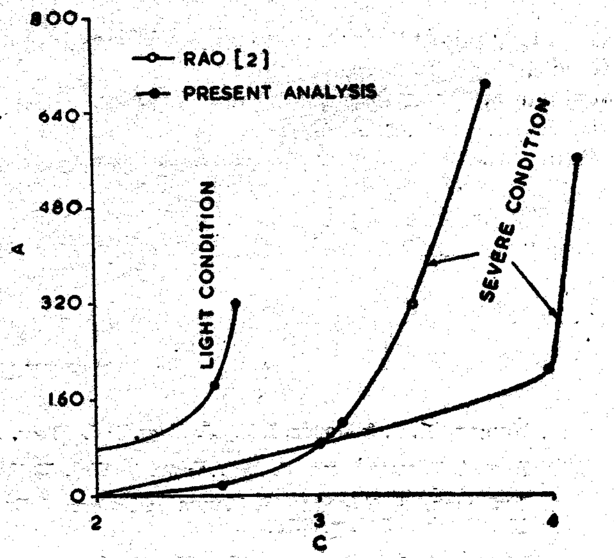

FIG. 4—Spring index for minimum volume.

$$= \frac{5.28 \times 8100 \times 1{\cdot}253 \times 15}{2 \times 36 \times 6 \times 40} = 46{\cdot}5 .$$

Total number of coils $= n + i = 47{\cdot}5$.

Example II

In addition to conditions specified in Example I, design the spring for minimum weight.

Design Constant :

$$A = \frac{G\,\delta}{2\,Pi} \left(\frac{8\,P}{\pi\,\phi} \right)^{\frac{1}{2-m}}$$

$$= \frac{8100 \times 15}{2 \times 40 \times 1} \left(\frac{8 \times 40}{\pi \times 40} \right)^{\frac{1}{1\cdot78}}$$

Therefore $\quad A = 2580$

$$\log A = 3{\cdot}4116$$

The value of C is obtained from the Fig. 3

$$C = 10{\cdot}1$$

Putting this value in (7), we get the diameter.

$$d = \left(\frac{8 \times 1{\cdot}27 \times 40 \times 10{\cdot}1}{\pi \times 40} \right)^{\cdot562} = 7{\cdot}07.$$

Putting this value in (18), we get

$$n = \frac{5{\cdot}08 \times 8100 \times 1{\cdot}27 \times 15}{2 \times (10{\cdot}1)^3 \times 40} = 133.$$

Total number of coils $= n + i = 134$.

DISCUSSION

The present design concerns with conical helical springs, torsion springs and leaf springs. For conical springs it is shown that the value of spring index is greater than that for a corresponding cylindrical spring for a given value of parameter A, for the design of spring for minimum weight. It can be observed from (Fig. 4) that in the case of optimization with respect to volume the conical spring gives a lower spring index for the same value of A.

The problem solved in the paper shows that for the same data (*i. e.* load, deflection and material of the spring) the number of coils needed to satisfy the requirements increases when the design is optimized.

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Appendix

A conical spring is shown in Fig. 5. Consider an element of this spring at a distance x from the apex of its cone. The length dL of this element is

$$dL = \pi D_x \frac{(n+i)}{H-h} dx \qquad (1)$$

where $D_x = \frac{x}{H} D_2$.

Therefore, the length L of the spring is

$$L = \int_h^H \pi \frac{x}{H} D_2 \frac{(n+i)}{H-h} dx .$$

$$= \frac{\pi D_2 (n+i)}{H(H-h)} \int_h^H x dx$$

$$= \frac{\pi D_2 (n+i)}{2H} (H+h)$$

Hence

$$W = \rho \frac{\pi D_2}{2H} (H+h)(n+i) \frac{\pi}{4} d^2 . \qquad (2)$$

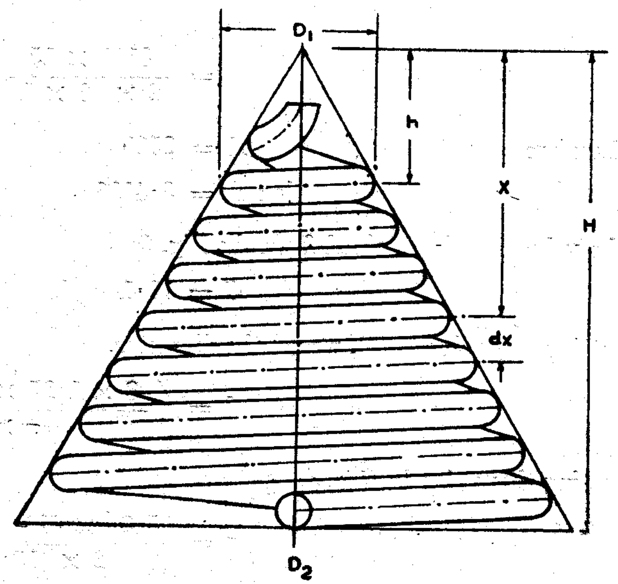

FIG. 5—Conical spring.