A STUDY OF FRACTURE IN A ROTATING DISC

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This investigation deals with the fracture produced in a disc by simple rotation cycles as well as rotation cycles under an imposed hydrostatic pressure. In both the cases, relations are obtained between the critical angular speed ω and the number of cycles N, required to cause fracture.

NOTATIONS

τ_1, τ_2, τ_3—Major, intermediate, minor principal elastic stresses

k, K —Slip moduli of the material

α, ω —Fatigue moduli of the material

e_r, e_θ, e_z—Principal plastic strains

σ_r, σ_θ, σ_z—Principal plastic stresses

σ_1, σ_2, σ_3—Major, intermediate, minor principal plastic stresses

S_r, S_θ, S_z—Plastic stress deviators

BASIC ASSUMPTION

The theory of fracture of brittle solids developed by T.Y. Thomas[1,2] is applied to a thin solid disc rotating about an axis through the centre perpendicular to its plane. The fracture is produced in the disc by simple rotation cycles as well as rotation cycles under an imposed hydrostatic pressure.

(*i*) The material of the disc is brittle. It undergoes an abrupt change from its elastic state to a state of plastic equilibrium, immediately prior to fracture. It fractures without the necessity of increasing the angular speed beyond the point required to produce the initial plastic deformation.

(*ii*) A simple rotation cycle is one in which the angular speed increases continually in numerical value from zero to a maximum value and then decreases continually to the initial value zero. The maximum numerical value of ω, of a rotation cycle, is called the peak load of the cycle. The peak load at which fracture occurs is called the critical load of the cycle. In this paper only the critical values of ω which depend on the number of cycles N have been used. It is a constant for a given value of N.

(*iii*) A function is defined as

$$\Omega(\tau) = (\tau_1 - \tau_3) - 2\bar{\omega} \tag{1}$$

(*iv*) If fracture is produced during the Nth cyclic load, the equation

$$(\sqrt{1+k^2}+k)\,\sigma_1 - (\sqrt{1+k^2}-k)\,\sigma_3 = 2K - 2\alpha(N-1)\,\Omega(\tau) \tag{2}$$

will be satisfied by the stress field in the plastic state preceding fracture. If $N = 1$, equation (2) reduces to

$$(\sqrt{1+k^2}+k)\,\sigma_1 - (\sqrt{1+k^2}-k)\,\sigma_3 = 2K \tag{3}$$

which gives the fracture problem for a single loading operation.

(*v*) The moduli k, $\bar{\omega}$ and K satisfy the inequality

$$(\sqrt{1+k^2}+k)\,\bar{\omega} < K \tag{4}$$

where k is very small and K is very large when measured in conventional units.

(*vi*) The disc is assumed to be so thin that the condition of plane stress exists.

(*vii*) *Endurance limit* : The magnitude L of the peak load satisfies the inequality $L_1 < L \leqslant L_2$ for loads which are just sufficient to cause fracture during the Nth loading operation. The upper limit L_2 is the magnitude of the peak load when fracture is produced by a single loading operation; the lower limit L_1 is referred to as the endurance limit.

FORMULATION

The cylindrical coordinates r, θ, z are employed such that the z-axis lies along the axis of rotation of the disc. It is assumed that there are no external forces acting on the disc and that it is of constant thickness.

In addition to the condition of fatigue fracture (2), the following relations are satisfied by the non-vanishing stress and strain fields induced in the disc, prior to fracture:

Equation of incompressibility

$$e_r + e_\theta + e_z = 0 \tag{5}$$

Hencky stress-strain relations

$$S_r = \psi\, e_r \;;\quad S_\theta = \psi\, e_\theta \;;\quad S_z = \psi\, e_z \tag{6}$$

Equation of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0 \tag{7}$$

where ρ is the density of the material.

The proportionality factor ψ is assumed to be a positive function and is regarded as one of the dependent variables of the problem.

The boundary condition of the problem is

$$\sigma_r = 0 \quad \text{at } r = a \tag{8}$$

where r is the distance of any point of the disc from its centre and 'a', its radius.

ANALYSIS

Following Hoffman and Sachs[3], the major, intermediate and minor principal plastic stresses are respectively

$$\sigma_1 = \sigma_\theta \,, \quad \sigma_2 = \sigma_r, \quad \sigma_3 = 0$$

Substituting these values of σ_1 and $\sigma_{3\theta}$ in (2), we have

$$(\sqrt{1 + k^2} + k)\, \sigma_\theta = 2K - 2\,\alpha\,(N - 1)\,\Omega\,(\tau) \tag{9}$$

The principal elastic stresses[3] are

$$\tau_1 = \frac{\rho\omega^2}{16}\,(7\,a^2 - 5\,r^2)$$

$$\tau_2 = \frac{7\,\rho\omega^2}{16}\,(a^2 - r^2)$$

$$\tau_3 = 0$$

where $\quad \tau_1 \geqslant \tau_2 \geqslant \tau_3$

Using these values in (1), we find that

$$\Omega\,(\tau) = \frac{\rho\omega^2}{16}\,(7\,a^2 - 5\,r^2) - 2\,\bar{\omega} \tag{10}$$

and hence equation (9) becomes

$$(\sqrt{1+k^2}+k)\,\sigma_\theta = 2K - 2\alpha(N-1)\left[\frac{\rho\omega^2}{16}(7a^2-5r^2) - 2\bar{\omega}\right] \tag{11}$$

From plane stress and symmetry considerations σ_r and σ_θ are functions of r only. Therefore equation (7) reduces to the form

$$\frac{d\sigma_r}{dr} - \frac{1}{r}(\sigma_\theta - \sigma_r) + \rho\omega^2 r = 0 \tag{12}$$

Using the value of σ_θ from (11) in the above and solving, we get

$$\sigma_r = \frac{2K}{\sqrt{1+k^2}+k} - \frac{2\alpha(N-1)}{\sqrt{1+k^2}+k}\left[\frac{\rho\omega^2}{16}\left(7a^2 - \frac{5r^2}{3}\right) - 2\bar{\omega}\right] - \frac{\rho\omega^2 r^2}{3} \tag{13}$$

on taking the constant of integration to be zero to avoid σ_r becoming infinite at $r = 0$.

On using the boundary condition (8) in (13), we find that

$$\omega^2 = \frac{6\,[K + 2\alpha\bar{\omega}(N-1)]}{\rho a^2\,[(\sqrt{1+k^2}+k) + 2\alpha(N-1)]} \tag{14}$$

which gives the critical angular speed ω, prior to fracture as a function of N.

Equation (14) can be written as

$$\left(\frac{\rho M a^2}{3} - 2\bar{\omega}\right)\left[(\sqrt{1+k^2}+k) + 2\alpha(N-1)\right] = 2\left[K - \bar{\omega}(\sqrt{1+k^2}+k)\right] \tag{15}$$

where $M = \omega^2$. It represents a hyperbola in the variables M and N. The second factor on the left side of equation (15) is always positive. Also the right hand member is positive because of (4). Hence

$$\frac{\rho M a^2}{3} - 2\bar{\omega} > 0$$

Therefore the endurance limit L_1 has the value

$$L_1 = M^{\frac{1}{2}} = \frac{1}{a}\left(\frac{6\bar{\omega}}{\rho}\right)^{\frac{1}{2}}$$

From the relation (14), we can deduce the value of ω which will cause fracture in a single loading operation. On setting $N = 1$ in (14), we observe that

$$\omega^2 = \frac{6K}{\rho a^2(\sqrt{1+k^2}+k)} \tag{16}$$

The nature of the graph of equation (15) for $N \geqslant 1$ is shown in Fig. 1. The endurance limit L is given

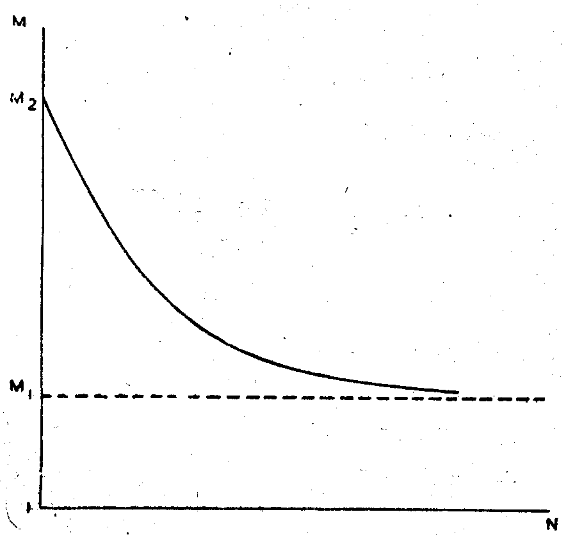

Fig. 1*—Graph of simple rotation cycles.

$^*M_1 = \frac{6\bar{\omega}}{\rho a^2}$ = Square of the endurance limit, L_1

$M_2 = \frac{6K}{\rho a^2(\sqrt{1+k^2}+k)}$ = Square of the magnitude of peak load, L_2, when fracture is produced by single loading by the asymptote $M = \frac{6\,\bar{\omega}}{\rho\,a^2}$ of the hyperbola. Also it is seen from Fig. 1 that cycles with $M \leqslant \frac{6\,\bar{\omega}}{\rho a^2}$ will not cause fracture in the material.

By setting $N = 1$, in equations (9) and (13) and using the value of ω^2 from (16), we obtain the following stres field preceding fracture in the case of a single loading operation:

$$\sigma_r = \frac{2\,K}{\sqrt{1+k^2}+k}\left(1 - \frac{r^2}{a^2}\right),$$

$$\sigma_\theta = \frac{2\,K}{\sqrt{1+k^2}+k}$$

DISC ROTATING UNDER HYDROSTATIC PRESSURE

Denoting the hydrostatic pressure by p and the resulting stress field by σ', we have

$$\sigma_1 = \sigma'_\theta = \sigma_\theta - p\,,$$
$$\sigma_2 = \sigma'_r = \sigma_r - p\,,$$
$$\sigma_3 = \sigma'_z = -\,p$$

where

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$$

On substituting these values for σ_1 and σ_3 and for $\Omega\,(\tau)$ from equation (10) in (2), we get after simplification

$$\sigma'_\theta = \frac{2\,K - p\,(\sqrt{1+k^2}-k)}{\sqrt{1+k^2}+k} - \frac{2\,\alpha\,(N-1)}{\sqrt{1+k^2}+k}\left[\frac{\rho\omega^2}{16}\,(7\,a^2 - 5\,r^2) - 2\,\bar{\omega}\right] \qquad (17)$$

Equation of equilibrium (12) now becomes

$$\frac{d\sigma'_r}{dr} - \frac{1}{r}\left(\sigma'_\theta - \sigma'_r\right) + \rho\omega^2 r = 0 \qquad (18)$$

Using the value of σ'_θ in the above equation and integrating, we get

$$\sigma'_r = \frac{2\,K - p\,(\sqrt{1+k^2}-k)}{\sqrt{1+k^2}+k} - \frac{2\,\alpha\,(N-1)}{\sqrt{1+k^2}+k}\left[\frac{\rho\omega^2}{16}\left(7\,a^2 - \frac{5\,r^2}{3}\right) - 2\,\bar{\omega}\right] - \frac{\rho\omega^2 r^2}{3} \qquad (19)$$

after omitting the constant of integration so as to avoid σ'_r becoming infinite at $r = 0$.

On using the boundary condition $\sigma'_r = -\,p$ at $r = a$ in (19), the critical speed preceding fracture is given by

$$\omega^2 = \frac{6}{\rho\,a^2}\left[\frac{(K + k\,p) + 2\,\alpha\,\bar{\omega}\,(N-1)}{(\sqrt{1+k^2}+k) + 2\,\alpha\,(N-1)}\right] \qquad (20)$$

Or, we have

$$\left(\frac{\rho\,M\,a^2}{3} - 2\,\bar{\omega}\right)\left[(\sqrt{1+k^2}+k) + 2\,\alpha\,(N-1)\right]$$
$$= 2\left[K - \bar{\omega}\,(\sqrt{1+k^2}+k)\right] + 2\,k\,p \qquad (21)$$

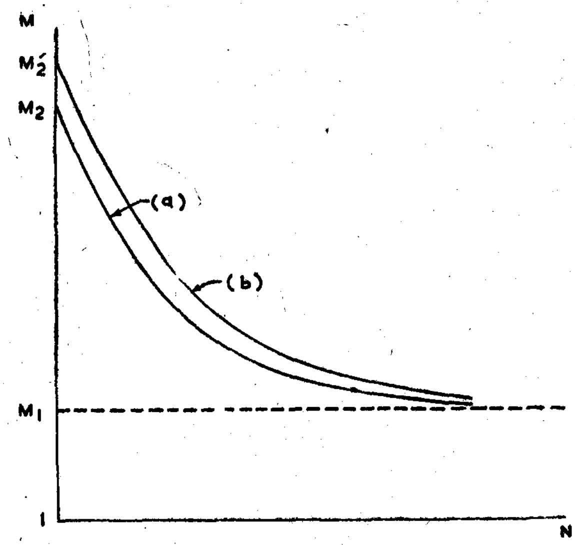

Fig. 2*—(a) Graph of simple rotation cycles
(b) Graph of simple rotation cycles under hydrostatic pressure .

where $M = \omega^2$. The second factor on the left side and the second term on the right side of equation (21) are always positive. Also the first term on the right side is positive because of (4). Hence

$$\frac{\rho M a^2}{3} - 2 \bar{\omega} > 0$$

Therefore the endurance limit L_1 has the value

$$L_1 = M^{\frac{1}{2}} = \frac{1}{a} \left(\frac{6 \bar{\omega}}{\rho} \right)^{\frac{1}{2}}$$

As before on setting $N = 1$, in equation (20), we get

$$\omega^2 = \frac{6 (K + kp)}{\rho a^2 (\sqrt{1 + k^2} + k)} \qquad (22)$$

which gives the angular speed necessary to cause fracture in a single loading operation under hydrostatic pressure.

The nature of the graphs of the hyperbolas (15) and (21) for $N \geqslant 1$ are shown in Fig. (2) for the convenience of comparison.

It is clear from Fig. 2 that, for a given number of cycles, the load required to cause fracture under hydrostatic pressure must be larger than the load required under atmospheric pressure. For a single loading operation, this result can be verified by comparing the two values of ω^2 obtained in equations (16) and (22).

On setting $N = 1$, in equations (17) and (19) and using the value of ω^2 from (22), we obtain the following stress field prior to fracture when the disc rotates under an imposed hydrostatic pressure, in the case of a single loading operation :

$$\sigma'_r = \frac{2K}{\sqrt{1 + k^2} + k} \left(1 - \frac{r^2}{a^2} \right) - \frac{p}{\sqrt{1 + k^2} + k} \left[(\sqrt{1 + k^2} - k) + \frac{2kr^2}{a^2} \right]$$

$$\sigma'_\theta = \frac{2K - p(\sqrt{1 + k^2} - k)}{\sqrt{1 + k^2} + k}$$

$$\sigma'_z = -p$$

The above work has tremendous potentialities of practical application because of its close relationship to the problem of ship propellers which become brittle due to passage of time under exposed and marine conditions. The propellers can be rotated repeatedly without danger of collapse due to fracture provided the angular speed ω is always such that $\omega \leqslant \frac{1}{a} \left(\frac{6 \bar{\omega}}{\rho} \right)^{\frac{1}{2}}$. Another possible application would be in the design of rotor discs in turbo-prop aircrafts which rotate under high velocities and pressure.

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*M_1, M_2 have the same values and significance as in Fig. 1

$M'_2 = \frac{6 (K + kp)}{\rho a^2 (\sqrt{1 + k^2} + k)}$ = Square of the magnitude of peak load when fracture is produced by single loading under hydrostatic pressure.

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