THE STABILITY OF DISSIPATIVE COUETTE FLOW IN HYDROMAGNETICS

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(*Received* 16 *August* 1972; *revised* 9 *August* 1973)

The stability of dissipative Couette flow in the presence of an axial volume current superposed by an axial uniform magnetic field parallel to the axis of the rotating column has been studied. The critical Taylor numbers for certain wave numbers have been obtained. It is found that the critical Taylor numbers at which the instability sets-in are increased.

The Couette flow with a pressure gradient is identical to flow of lubricating oil in the narrow space in between journal and bearing, which is observed in all moving parts of aircrafts. It can be used in making and testing the models of ships and submarines. Before building new type of ship, its models are made and tested experimentally.

Chandrasekhar[1] has studied the stability of non-dissipative Couette flow in presence of an axial and a transverse magnetic field. In the present paper, the problem of dissipative Couette flow in the presence of an axial volume current superposed by an axial uniform magnetic field parallel to the axis of the rotating column has been discussed. The problem has been restricted to the axisymmetric perturbations and small gap approximation. The critical Taylor numbers have been obtained numerically for different wave numbers.

BASIC EQUATIONS

Consider the flow of an incompressible, viscous, electrically conducting fluid between two concentric rotating cylinders in the presence of an axial volume current and an axial uniform magnetic field. Following Chandrasekhar[2], it is observed that the basic equations of hydromagnetic allow the stationary solutions.

$$U_r = U_z = 0\,, \quad U_\theta = V = Ar + B/r\,,$$

$$H_r = 0\,, \quad H_z = H_0\,, \quad H_\theta = H\theta\,(r) = \Omega\, r\,, \tag{1}$$

where A and B are two constants. These constants in (1) are related to the angular velocities Ω_1 and Ω_2 of the two cylinders confining the fluid. They are given by

$$A = -\,\Omega_1\, \nu_H^2\, \frac{1-\mu/\nu_H^2}{1-\nu_H^2}\,, \; B = \Omega_1\, \frac{R_1^2\,(1-\mu)}{1-\nu_H^2}\,,$$

$$\mu = \frac{\Omega_2}{\Omega_1}\,, \quad \nu_H = \frac{R_1}{R_2}\,. \tag{2}$$

Ω_1, Ω_2, R_1, R_2 being the angular velocities and radii of inner and outer cylinders respectively. The total pressure P is given by the radial component of the momentum equation

$$-\frac{dP}{dr} = -\frac{V^2}{r} + \frac{\mu}{4\pi\rho}\left(\Omega^2 r + \frac{H_0^2}{r}\right). \tag{3}$$

These solutions correspond to the unperturbed state.

The perturbed state is described by

$$u_r\,,\; V+u_\theta\,,\; u_z\,,\; h_r\,,\; H_\theta+h_\theta\,,\; H_0+h_z\,,\; \varpi\,(=\delta\,\pi)\,. \tag{4}$$

The linearized axisymmetric equations are

$$\frac{\partial u_r}{\partial t} - \frac{2V}{r} u_\theta - \frac{\mu}{4\pi\rho}\left(H_0 \frac{\partial h_r}{\partial z} - \frac{2H_\theta h_\theta}{r}\right) = -\frac{\partial \varpi}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right), \quad (5)$$

$$\frac{\partial u_\theta}{\partial t} + (D_* V) u_r - \frac{\mu}{4\pi\rho}\left\{H_0 \frac{\partial h_\theta}{\partial z} + h_r (D_* H_\theta)\right\} = \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right), \quad (6)$$

$$\frac{\partial u_z}{\partial t} - \frac{\mu}{4\pi\rho} H_0 \frac{\partial h_z}{\partial_z} = -\frac{\partial \varpi}{\partial z} + \nu \nabla^2 u_z, \quad (7)$$

$$\frac{\partial h_r}{\partial t} = H_0 \frac{\partial u_r}{\partial z} + \nu_H\left(\nabla^2 h_r - \frac{h_r}{r^2}\right), \quad (8)$$

$$\frac{\partial h_\theta}{\partial t} = H_0 \frac{\partial u_\theta}{\partial z} + r h_r \frac{\partial}{\partial r}\left(\frac{V}{r}\right) - r u_r \frac{\partial}{\partial r}\left(\frac{H_\theta}{r}\right) + \nu_H\left(\nabla^2 h_\theta - \frac{h_\theta}{r^2}\right), \quad (9)$$

$$\frac{\partial h_z}{\partial t} = H_0 \frac{\partial u_z}{\partial z} + \nu_H \nabla^2 h_z, \quad (10)$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0, \quad (11)$$

$$\frac{\partial h_r}{\partial r} + \frac{h_r}{r} + \frac{\partial h_z}{\partial z} = 0, \quad (12)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \text{ and } D_* \equiv \frac{d}{dr} + \frac{1}{r}.$$

We assume that the first-order variations in all the physical quantities are of the form

$$f(r) \exp. (ikz + pt). \quad (13)$$

eqns. (5) to (12) reduce to

$$\nu\left(DD_* - k^2 - \frac{p}{\nu}\right) u_r + \frac{\mu H_0 ikh_r}{4\pi\rho} + \frac{2V}{r} h_\theta - \frac{2\mu H_\theta h_\theta}{r} = D\varpi, \quad (14)$$

$$\nu\left(DD_* - k^2 - \frac{p}{\nu}\right) u_\theta + \frac{\mu H_0 ik h_\theta}{4\pi\rho} - (D_* V) u_r + \frac{\mu}{4\pi\rho}\left(\frac{2H_\theta h_r}{r}\right) = 0, \quad (15)$$

$$\nu\left(D_* D - k^2 - \frac{p}{\nu}\right) u_z + \frac{\mu H_0 ikh_z}{4\pi\rho} = ik\varpi, \quad (16)$$

$$\nu_H\left(DD_* - k^2 - \frac{p}{\nu_H}\right) h_r = -H_0 iku_r, \quad (17)$$

$$\nu_H\left(DD_* - k^2 - \frac{p}{\nu_H}\right) h_\theta + \left(\frac{dV}{dr} - \frac{V}{r}\right) h_r = -H_0 iku_\theta, \quad (18)$$

$$\nu_H\left(D_* D - k^2 - \frac{p}{\nu_H}\right) h_z = -H_0 iku_z, \quad (19)$$

$$D_* u_r = -iku_z; \quad D_* h_r = -ikh_z. \quad (20)$$

From (14), (16) and (20) we obtain

$$\left(DD_* - k^2 - \frac{p}{\nu}\right)(DD_* - k^2) u_r + \frac{\mu H_0 ik}{4\pi\rho\nu}(DD_* - k^2) h_r + \frac{2\mu k^2 H_\theta h_\theta}{4\pi\rho\nu r} = \frac{2Vk^2 u_\theta}{r\nu}. \quad (21)$$

eqns. (15), (17), (18) and (21) are rewritten as

$$(DD_* - a^2 - \sigma) u_\theta + \frac{\mu H_0 idah_\theta}{4\pi\rho\nu} + \frac{2\mu\Omega d^2 h_r}{4\pi\rho\nu} = \frac{2Ad^2 u_r}{\nu}, \quad (22)$$

$$(DD_* - a^2 - \epsilon\sigma)\, h_r = - \frac{H_0\, iadu_r}{\nu_H}\, , \tag{23}$$

$$(DD_* - a^2 - \epsilon\sigma)\, h_\theta = \frac{2\, B}{r^2}\, \frac{d^2 h_r}{\nu_H} - \frac{H_0\, idau_\theta}{\nu_H}\, , \tag{24}$$

$$(DD_* - a^2 - \sigma)\ (DD_* - a^2)\, u_r + \frac{\mu i H_0\, da}{4\, \pi\rho\nu}(DD_* - a^2)\, h_r + \frac{2\, \mu a^2 d^2\, \Omega h_\theta}{4\, \pi\rho\nu} =$$

$$= \frac{2\, a^2 d^2}{\nu}\ \Omega_1 \Big[1 - (1 - \mu)\, \zeta \Big] u_\theta\, , \tag{25}$$

where

$$\zeta = \frac{r - R_1}{d}, \quad k = \frac{a}{d}\, , \quad \mu = \frac{\Omega_2}{\Omega_1}, \quad d = R_2 - R_1, \quad \sigma = \frac{pd^2}{\nu}, \quad \epsilon = \frac{\nu}{\nu_H}. \tag{26}$$

If the gap is narrow, (24) reduces to

$$(D^2 - a^2 - \epsilon\sigma)\ h_\theta = - \frac{H_0\, adiu_\theta}{\nu_H}\, . \tag{27}$$

From (22) with the help of (23) and (27) we obtain

$$\Big[(D^2 - a^2 - \epsilon\sigma)\ (D^2 - a^2 - \sigma) + Qa^2 \Big] (D^2 - a^2 - \epsilon\sigma)\, h_\theta =$$

$$= - i \Big[(D^2 - a^2 - \epsilon\sigma) + \frac{QR}{A}\ i \Big] u_r\, , \tag{28}$$

where

$$Q = \frac{\mu H_0^{\,2} d^2}{4\, \pi\rho\, \nu\, \nu_H}, \quad R = \frac{\Omega\, a\nu}{H_0\, d}\, . \tag{29}$$

and $\dfrac{H_0\, da}{\nu_H} \cdot \dfrac{2\, Ad^2\, h_\theta}{\nu}$ is replaced by h_θ.

Again from (25) using (23) and (27), we get

$$\Big[(D^2 - a^2 - \epsilon\sigma)\ (D^2 - a^2 - \sigma) + Qa^2 \Big] (D^2 - a^2)\, u_r$$

$$= - iTa^2 \Big[(D^2 - a^2 - \epsilon\sigma)\, (1 - \overline{1 - \mu}\, \zeta) + \frac{iQR}{\Omega_1} \Big] (D^2 - a^2 - \epsilon\sigma)\, h_\theta\, , \tag{30}$$

where $T = - \dfrac{4\, A\, \Omega_1\, d^4}{\nu^2}$ is the Taylor number for narrow gaps.

SOLUTION OF THE CHARACTERITIC VALUE PROBLEM FOR THE CASE $\mu > 0$ AND $\sigma = 0$

Let

$$\bar{T} = \frac{1}{2}\, (1 + \mu)\, T\, , \tag{31}$$

and

$$G = (D^2 - a^2)\, h_\theta\, . \tag{32}$$

The equations to be solved are

$$\Big[(D^2 - a^2)^2 + Qa^2 \Big] (D^2 - a^2)\, h_\theta = - i \Big[(D^2 - a^2) + iL \Big] u_r \tag{33}$$

and

$$\left[(D^2 - a^2)^2 + Qa^2\right](D^2 - a^2)u_r = -i\bar{T}a^2\left[(D^2 - a^2) + iN\right]G , \tag{34}$$

where

$$L = \frac{QR}{A} , \quad N = \frac{2QR}{(1+\mu)\,\Omega_1} . \tag{35}$$

A Variational Principle

The problem presented by (33) and (34) with the boundary conditions can be formulated in terms of a variational principle. After multiplying (34) by $-i\left[(D^2 - a^2) + iL\right]u_r$ and integrating over the range of ζ we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\left\{(D^2 - a^2)^2 + Qa^2\right\}(D^2 - a^2)u_r \left\{(D^2 - a^2) + iL\right\} u_r\right] d\zeta$$

$$= -\bar{T}a^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\left\{(D^2 - a^2) + iN\right\} G \left\{(D^2 - a^2)^2 G + Qa^2 G\right\}\right] d\zeta . \tag{36}$$

After one or more integration by parts (in which the integrated parts vanish on account of boundary conditions), we find that both sides of (36) can be brought to positive definite forms and the result is

$$\left.\begin{aligned}\bar{T} = \frac{\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[(D^4 u_r)^2 + 4a^2 (D^3 u_r)^2 + 6a^4 (D^2 u_r)^2 + 4a^6 (Du_r)^2 + iL \{(D^2 - a^2)^2 + Qa^2\} \times \{(D^2 - a^2)u_r\} u_r \Big] d\zeta}{\displaystyle a^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[(D^3 G^2)^2 + 2a^2 (D^2 G)^2 + a^2 (3a^2 + Q + 1) (DG)^2 + a^4 (Q + a^2) G^2 + iN \{DGD^3 G + 2a^2 (DG)^2 + Dh_\theta DG + a^2 h_\theta G\} \Big] d\zeta}\end{aligned}\right\} . \tag{37}$$

The characteristic values of $\bar{T}$ which is a certain ratio of two positive definite integrals, represent the extremal values as shown by (37).

The critical Taylor number $\bar{T}$ for the onset of instability (for a given Q) represents the absolute minimum.

Here (33) and (34) are solved under the conditions (i) $\sigma = 0$, (ii) $\mu > 0$. When $\sigma = 0$, the marginal state is stationary and for $\mu > 0$, it is established that the occurrence of overstability is effectively excluded [Chandrasekhar[3]]. In both the cases, at the onset of instability a stationary pattern of motion prevails, this indicates that the principle of exchange of stability is valid. If $\mu < 0$, the governing eqns. (33) and (34) will lead to rapidly increasing errors and the possibility of overstability occurring cannot be excluded. But as we are considering the solutions for $\mu > 0$ so the overstability would seem unlikely.

The standard forms of these solutions are taken in the form of two orthogonal functions[2].

$$C_m(x) = \frac{\cosh \lambda_m x}{\cosh \frac{1}{2} \lambda_m} - \frac{\cos \lambda_m x}{\cos \frac{1}{2} \lambda_m} , \tag{38}$$

$$S_m(x) = \frac{\sinh \mu_m x}{\sinh \frac{1}{2} \mu_m} - \frac{\sin \mu_m x}{\sin \frac{1}{2} \mu_m} , \tag{39}$$

where λ_m and μ_m ($m = 1, 2, 3, \ldots\ldots$) are the positive roots of the eqns.

$$\tanh \frac{1}{2} \lambda + \tan \frac{1}{2} \lambda = 0 \text{ and } \coth \frac{1}{2} \mu - \cot \frac{1}{2} \mu = 0 . \tag{40}$$

The functions $C_m(x)$ and $S_m(x)$ satisfy the orthogonality relations.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(x)\; C_n(x)\; dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_m(x)\; S_n(x)\; dx = \delta_{mn} \tag{41}$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(x)\; S_n(x)\; dx = 0 \,. \tag{42}$$

Considering only even solutions, we expand u_r in terms of the functions C_m ; thus

$$u_r = \Sigma_m\, A_m\, C_m(\zeta)\,, \tag{43}$$

where the summation over m may be considered as running from 1 to ∞. The corresponding expansion for h_θ is given by

$$h_\theta = \Sigma_m\, A_m\, h_\theta{}^m\;(\zeta)\,, \tag{44}$$

where $h_\theta{}^m(\zeta)$ is the solution of

$$[(D^2 - a^2)^2 + Qa^2]\,(D^2 - a^2)\, h_\theta{}^m(\zeta) = -\,i\,[(D^2 - a^2) + iL]\, C_m(\zeta) \tag{45}$$

which satisfy the boundary condition on $h_\theta{}^m$.

The general solution of (45) which is even can be written in the form

$$h_\theta{}^m = \frac{(ia^2 + L - i\,\lambda_m{}^2)}{[(\lambda_m{}^2 - a^2)^2 + Qa^2]\,(\lambda_m{}^2 - a^2)}\;\frac{\cosh \lambda_m\,\zeta}{\cosh \frac{1}{2}\,\lambda_m} + \frac{(ia^2 + L + i\lambda_m{}^2)\cos \lambda_m\,\zeta}{[(\lambda_m{}^2 + a^2) + Qa^2]\,(\lambda_m{}^2 + a^2)\cos\frac{1}{2}\,\lambda_m} +$$

$$+\,\beta_1{}^{(m)}\cosh q_1\,\zeta + \beta_2{}^{(m)}\cosh q_2\,\zeta + \beta_3{}^{(m)}\cosh q_3\,\zeta\,, \tag{46}$$

where $\beta_1{}^{(m)}$, $\beta_2{}^{(m)}$, $\beta_3{}^{(m)}$ are constants of integration and

$$q_1{}^2 = a^2 + ia\sqrt{Q}\,,\; q_2{}^2 = a^2 - ia\sqrt{Q}\,,\; q_3{}^2 = a^2 \tag{47}$$

are the roots of

$$[(q^2 - a^2)^2 + Qa^2]\,(q^2 - a^2) = 0\,. \tag{48}$$

Without loss of generality, we may write

$$q_1 = \alpha_1 + i\alpha_2\,,\; q_2 = \alpha_1 - i\alpha_2\,, \tag{49}$$

where

$$\alpha_1 = \left[\frac{1}{2}\sqrt{a^4 + Qa^2} + \frac{a^2}{2}\right]^{\frac{1}{2}},\; \alpha_2 = \left[\frac{1}{2}\sqrt{a^4 + Qa^2} - \frac{a^2}{2}\right]^{\frac{1}{2}}. \tag{50}$$

It is apparent that $\beta_1{}^{(m)}$ and $\beta_2{}^{(m)}$ are complex conjugates.

An identity which follows from (48) is

$$\Gamma_m = \frac{1}{|\,\lambda_m{}^4 - q_1{}^4\,|^2\,|\,\lambda_m{}^4 - q_3{}^4\,|}\,. \tag{51}$$

Using (47) we obtain

$$(\lambda_m{}^4 - q_{1,2}{}^4)\;(\lambda_m{}^4 - q_3{}^4) = (\lambda_m{}^4 - a^4 + a^2\,Q \mp 2\,ia^3\,\sqrt{Q})\;(\lambda_m{}^4 - a^4)\,. \tag{52}$$

If we let

$$g_m = \frac{1}{a\sqrt{Q}}\;(\lambda_m{}^4 - a^4 + Qa^2)\,, \tag{53}$$

Then

$$(\lambda_m{}^4 - q_{1,2}{}^4) = (g_m \mp 2\,ia^2)\,a\sqrt{Q}\;(\lambda_m{}^4 - a^4)\,. \tag{54}$$

An alternative expression for Γ_m is given by

$$\frac{1}{\Gamma_m} = (g_m{}^2 + 4\,a^4)\,a^2\,Q\,(\lambda_m{}^4 - a^4)\,. \tag{55}$$

With the help of the foregoing relations and definitions, the solution (46) can be rewritten for h_θ^m in the form

$$h_\theta^m = \Gamma_m [A' C_m(\zeta) + B' C_m''(\zeta)] + \beta_1^{(m)} \cosh q_1 \zeta + \beta_2^{(m)} \cosh q_2 \zeta + \beta_3^{(m)} \cosh q_3 \zeta , \quad (56)$$

where

$$A' = i \, [a^6 (a^2 + Q) - \lambda_m^4 (Qa^2 + \lambda_m^4)] + L (3 a^2 \lambda_m^4 + a^6 + Qa^4) , \quad (57)$$

$$B' = 2 a^2 i (a^4 - \lambda_m^4) + L (\lambda_m^4 + 3 a^4 + Qa^2) ,$$

$$C_m''(\zeta) = \frac{d^2 C_m}{d \zeta^2} = \lambda_m^2 \left(\frac{\cosh \lambda_m (\zeta)}{\cosh \frac{1}{2} \lambda_m} + \frac{\cos \lambda_m (\zeta)}{\cos \frac{1}{2} \lambda_m} \right) . \quad (58)$$

With the help of (47) we obtain G_m from (56)

$$G_m = (D^2 - a^2) h_\theta^m$$

$$= \Gamma_m [(B' \lambda_m^4 - a^2 A') C_m(\zeta) + (A' - a^2 B') C_m''(\zeta)] + ia\sqrt{Q} (\beta_1^{(m)} \cosh q_1 \zeta - \beta_2^{(m)} \cosh q_2 \zeta) . \quad (59)$$

We observe that

$$[(D^2 - a^2)^2 + Qa^2] (D^2 - a^2) C_m(\zeta) = (\lambda_m^4 + 3 a^4 + Qa^2) C_m''(\zeta) - a^2 (3 \lambda_m^4 + a^4 + Qa^2) C_m(\zeta) . \quad (60)$$

Substituting the expansion for u_r and h_θ^m in (34) we obtain

$$\frac{1}{\overline{T} a^2} \left[\Sigma_m A_m \left\{ (\lambda_m^4 + 3 a^4 + Qa^2) C_m''(\zeta) - a^2 (3 \lambda_m^4 + a^4 + Qa^2) C_m(\zeta) \right\} \right] + i \, \Sigma_m A_m \Gamma_m \times$$

$$\times \left[\left\{ A' (\lambda_m^4 + a^4 - iNa^2) + B' \lambda_m^4 (iN - 2 a^2) \right\} C_m(\zeta) + \left\{ B' (\lambda_m^4 + a^4 - iNa^2) + \right. \right.$$

$$\left. \left. + A' (iN - 2 a^2) \right\} C_m''(\zeta) \right] - ia\sqrt{Q} \, \beta_1^{(m)} \cosh q_1 \zeta (a\sqrt{Q} + N) -$$

$$- ia\sqrt{Q} \, \beta_2^{(m)} \cosh q_2 \zeta (a\sqrt{Q} - N) = 0 . \quad (61)$$

Multiplying (61) by $C_n(\zeta)$ and integrating over the range ζ and making use of the orthogonality property of C-functions and with the further following definitions

$$X_{mn} = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_m''(\zeta) C_n(\zeta) \, d\zeta , \quad (62)$$

and

$$\left(\cosh q \zeta / C_n(\zeta) \right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_n(\zeta) \cosh q \zeta \, d\zeta . \quad (63)$$

We obtain the required secular determinant for $\overline{T}$

$$\Big\| \frac{1}{\overline{T} a^2} \left[(\lambda_m^4 + 3 a^4 + Q a^2) X_{mn} - a^2 (3 \lambda_m^4 + a^4 + Q a^2) \delta_{mn} \right] + \Gamma_m \left[i \left\{ A' (\lambda_m^4 + \right. \right.$$

$$\left. \left. + a^4 - iNa^2) + B' \lambda_m^4 (iN - 2 a^2) \right\} \delta_{mn} + i \left\{ B' (\lambda_m^4 + a^4 - iNa^2) + A' (iN - 2 a^2) \right\} X_{mn} \right] -$$

$$- ia\sqrt{Q} \left[(a\sqrt{Q} + N) \beta_1^{(m)} \left(\cosh q_1 \zeta / C_n(\zeta) \right) + (a\sqrt{Q} - N) \beta_2^{(m)} \left(\cosh q_2 \zeta / C_n(\zeta) \right) \right] \Big\| = 0 . \quad (64)$$

By elementary calculations we find

$$X_{mn} = \frac{2}{\lambda_m^4 - \lambda_n^4} (C_m''' C_n'' - C_n''' C_m'')_{\zeta = \frac{1}{2}} \; (m \neq n)$$

$$= \frac{1}{\lambda_n^4} \left[\frac{1}{2} C_n''' C_n'' - \frac{1}{4} (C_n''')^2 \right]_{\zeta = \frac{1}{2}} \; (m = n) \quad (65)$$

and

$$\left(\cosh q\zeta / C_n(\zeta) \right) = \frac{2}{\lambda_n^4 - q^4} \left[C_n'''(\tfrac{1}{2}) \cosh \tfrac{1}{2} q - C_n''(\tfrac{1}{2})\, q \sinh \tfrac{1}{2} q \right], \tag{66}$$

where

$$C_m''(\tfrac{1}{2}) = 2\lambda_m^2 \text{ and } C_m'''(\tfrac{1}{2}) = 2\lambda_m^3 \tanh \tfrac{1}{2}\lambda_m . \tag{67}$$

The last line of (64) has been simplified with the help of (51) and (53) and can be rewritten as

$$-ia\sqrt{Q}\left[(a\sqrt{Q}+N)\,\beta_1^{(m)} \left(\cosh q_1 \zeta / C_n(\zeta) \right) \right] + (a\sqrt{Q}-N)\,\beta_2^{(m)} \left(\cosh q_2 \zeta / C_n(\zeta) \right) =$$

$$= -4a^2 Q\, \Gamma_n (\lambda_n^4 - a^4) \left[\left\{ a\sqrt{Q}\, re\, (ig_n + 2a^2)\, \beta_1^{(m)} + N\, re\, (ig_n - 2a^2)\, \beta_1^{(m)} \right\} . \right.$$

$$\left. . \left(C_n'''(\tfrac{1}{2}) \cosh \tfrac{1}{2} q_1 - C_n''(\tfrac{1}{2})\, q_1 \sinh \tfrac{1}{2} q_1 \right) \right] \tag{68}$$

The second line of (64) has been simplified with the help of (57) and its real part can be rewritten as

$$\Gamma_m \left[i \left\{ A'(\lambda_m^4 + a^4 - iNa^2) + B'\lambda_m^4 (iN - 2a^2) \right\} \delta_{mn} + i \left\{ B'(\lambda_m^4 + a^4 - iNa^2) + \right. \right.$$

$$\left. \left. + A'(iN - 2a^2) \right\} X_{mn} \right] = \Gamma_m (K\,\delta_{mn} + M\,X_{mn}), \tag{69}$$

where

$$K = NL \left\{ a^6 (a^2 + Q) - \lambda_m^4 (Qa^2 + \lambda_m^4) \right\} + \lambda_m^8 (Qa^2 + \lambda_m^4) + 3a^4 \lambda_m^4 (a^4 -$$

$$- \lambda_m^4) - a^{10}(a^2 + Q), \tag{70}$$

$$M = 2a^2 (NL + a^2 Q)(a^4 - \lambda_m^4). \tag{71}$$

The secular eqn takes the form

$$\left|\left| \frac{1}{T a^2} \left[(\lambda_m^4 + 3a^4 + Qa^2) X_{mn} - a^2 (3\lambda_m^4 + a^4 + Qa^2)\, \delta_{mn} \right] + \Gamma_m (K\,\delta_{mn} + M\,X_{mn}) - \right.\right.$$

$$- 4a^2 Q\, \Gamma_n (\lambda_n^4 - a^4) \left[\left\{ a\sqrt{Q}\, re\, (ig_n + 2a^2)\, \beta_1^{(m)} + N\, re\, (ig_n - 2a^2)\, \beta_1^{(m)} \right\} . \right.$$

$$\left.\left.\left. . \, C_n'''(\tfrac{1}{2}) \cosh \tfrac{1}{2} q_1 - C_n''(\tfrac{1}{2})\, q_1 \sinh \tfrac{1}{2} q_1 \right] \right|\right| = 0 . \tag{72}$$

THE BOUNDARY CONDITIONS

The boundary conditions (1) $G_m = 0$, (2) $(D^2 - a^2)\ G_m = 0$, for $\zeta = \mp \frac{1}{2}$ are satisfied for both the cases (i) Non-conducting walls, (ii) Conducting walls. Using (59) we obtain

$$\beta_1^{(m)} \cosh \tfrac{1}{2} q_1 - \beta_2^{(m)} \cosh \tfrac{1}{2} q_2 = \frac{i\,\Gamma_m}{a\sqrt{Q}} \left[(A' - a^2 B')\, C_m''(\tfrac{1}{2}) \right],$$

$$\beta_1^{(m)} \cosh \tfrac{1}{2} q_1 + \beta_2^{(m)} \cosh \tfrac{1}{2} q_2 = \frac{\Gamma_m}{a^2 Q} \left[B'(\lambda_m^4 + a^4) - 2a^2 A' \right] C_m''(\tfrac{1}{2}), \tag{73}$$

where A' and B' are defined in (57).

From (73) we find $\beta_1^{(m)}$

$$\beta_1^{(m)} = \frac{\operatorname{sech}(\tfrac{1}{2})\, q_1\, \Gamma_m\, C_m''(\tfrac{1}{2})}{2a^2 Q} \left[A' a (i\sqrt{Q} - 2a) + B'(\lambda_m^4 + a^4 - i a^3 \sqrt{Q}) \right] \tag{74}$$

and $\beta_2^{(m)}$ is its complex conjugate.

With the help of (74), the last line of (72) takes the form

$$-2\,\Gamma_m \Gamma_n (\lambda_n^4 - a^4)(Z_{mn} + \Sigma_{mn}), \tag{75}$$

where
$$Z_{mn} = C_m''\left(\tfrac{1}{2}\right) C_n'''\left(\tfrac{1}{2}\right) \alpha , \tag{76}$$

$$\Sigma_{mn} = - C_m''\left(\tfrac{1}{2}\right) C_n''\left(\tfrac{1}{2}\right) \left[\alpha \, re \left(q_1 \tanh \tfrac{1}{2} q_1\right) + \gamma \, im \left(q_1 \tanh \tfrac{1}{2} q_1\right) \right] , \tag{77}$$

$$\alpha = 2 a^2 c_2 (a \sqrt{Q} - N) - g_n c_1 (a \sqrt{Q} + N) , \tag{78}$$

$$\gamma = 2 a^2 c_1 (a \sqrt{Q} - N) + g_n c_2 (a \sqrt{Q} + N) , \tag{79}$$

$$c_1 = 2 a^3 \sqrt{Q} (\lambda_m^4 - a^4) (L + a\sqrt{Q}) , \tag{80}$$

$$c_2 = \left[a^6 (a^2 - Q) + \lambda_m^4 (\lambda_m^4 + Q a^2 - 2 a^4) \right] (a \sqrt{Q} + L) . \tag{81}$$

Therefore, (72) takes the final form

$$\left|\left| \frac{1}{\bar{T} a^2} \left[(\lambda_m^4 + 3 a^4 + Q a^2) X_{mn} - a^2 (3 \lambda_m^4 + a^4 + Q a^2) \delta_{mn} \right] + \Gamma_m (K \delta_{mn} + M X_{mn}) - \right.\right.$$
$$\left.\left. - 2 \Gamma_m \Gamma_n (\lambda_n^4 - a^4) (Z_{mn} + \Sigma_{mn}) \right|\right| = 0 , \tag{82}$$

where K and M are defined in (70) and (71).

We may note that the real and imaginary parts of $q_1 \tanh \frac{1}{2} q_1$ which occur in the expression for Σ_{mn} are given by

$$re \left(q_1 \tanh \tfrac{1}{2} q_1\right) = \frac{\alpha_1 \sinh \alpha_1 - \alpha_2 \sin \alpha_2}{\cosh \alpha_1 + \cos \alpha_2} , \tag{83}$$

$$im \left(q_1 \tanh \tfrac{1}{2} q_1\right) = \frac{\alpha_2 \sinh \alpha_1 + \alpha_1 \sin \alpha_2}{\cosh \alpha_1 + \cos \alpha_2} , \tag{84}$$

where α_1 and α_2 are defined in (50).

NUMERICAL RESULTS

The (82) has been solved for a number of values of a and Q. The minimum value of the characteristic roots $\bar{T}$ has been determined for both the cases. The results of calculations are summarized in Table 1.

TABLE 1

CRITICAL TAYLOR NUMBERS AND RELATED CONSTANTS FOR DIFFERENT VALUES OF Q

If $\dfrac{A}{(1+\mu)\Omega_1} = -1$, then

Q	a	R/A	Second approximation	
			$\bar{T}$	Corresponding $\bar{T}$ of Chandrasekhar
5	3·20	−0·8893	$3\cdot3365 \times 10^5$	$2\cdot1853 \times 10^3$
10	3·30	−0·7007	$2\cdot9986 \times 10^5$	$2\cdot6924 \times 10^3$
20	3·40	−0·5023	$5\cdot0288 \times 10^5$	$3\cdot8093 \times 10^3$
50	3·45	−0·3702	$2\cdot9662 \times 10^5$	$7\cdot9926 \times 10^3$
100	3·35	−0·3608	$3\cdot5372 \times 10^5$	$1\cdot7573 \times 10^4$
30	2·68	−0·2364	$9\cdot2827 \times 10^5$	$3\cdot9657 \times 10^3$
100	1·69	$-0\cdot7187 \times 10^{-3}$	$4\cdot1858 \times 10^5$	$1\cdot0821 \times 10^4$

THE ASYMPTOTIC BEHAVIOUR FOR $Q \to \infty$

Let $$a^2 \to 0 \text{ while } Q a^2 \to a \text{ finite limit as } Q \to \infty . \tag{85}$$

On examining the original differential eqns (33) and (34), we have the following asymptotic behaviours:

$$Q a^2 \to Q_\infty \text{ and } \bar{T} a^2 \to T_\infty \text{ as } Q \to \infty \text{ and } a \to 0 . \tag{86}$$

The differential eqns. take the limiting forms

$$(D^4 + Q_\infty) D^2 h_\theta = - i (D^2 + i L') u_r , \tag{87}$$

$$(D^4 + Q_\infty) D^2 u_r = - i T_\infty (D^2 + i N') D^2 h_\theta , \tag{88}$$

where $$L' = Q_\infty R_\infty , \tag{89}$$

$$N' = 2 Q_\infty R_\infty \frac{A}{(1+\mu) \Omega_1} , \tag{90}$$

while the boundary conditions are unaffected and remain the same. To determine the correct asymptotic behaviours of the critical Taylor number and the associated wave number for $Q \to \infty$, we must solve (87) and (88) together with the proper boundary conditions. The problem can be solved exactly in the same manner of article (3) by putting $a^2 = 0$ in various expressions except when it occurs in the combinations of Qa^2 and Ta^2; they are then replaced by Q_∞ and T_∞ respectively. By defining the various quantities in the limit, (50), (53) and (55) become

$$\alpha_1 = \alpha_2 = \left(\frac{Q_\infty}{4} \right)^{\frac{1}{4}} , \; g_m = \frac{\lambda_m^4 + Q_\infty}{\sqrt{Q_\infty}} , \; \frac{1}{\Gamma_m} = g_m^2 Q_\infty \lambda_m^4 . \tag{91}$$

with these definitions, the limiting form of the secular eqn. (82) is

$$\left|\left| \frac{1}{\Gamma_\infty} (\lambda_m^4 + Q_\infty) X_{mn} + \frac{1}{g_m^2 Q_\infty} \left(\frac{K' \delta_{mn}}{\lambda_m^4} - \frac{2 \Sigma_{mn}}{g_n^2 Q_\infty \lambda_n^4} \right) \right|\right| = 0 , \tag{92}$$

where $$\Sigma_{mn} = - C_n'' (\tfrac{1}{2}) C_m'' (\tfrac{1}{2}) [\gamma' \, im \, (q_1 \tanh \tfrac{1}{2} q_1)] , \tag{93}$$

$$\gamma' = g_n c_2' (N' + \sqrt{Q_\infty}) , \tag{94}$$

$$c_2' = \lambda_m^4 (\lambda_m^4 + Q_\infty) (L' + \sqrt{Q_\infty}) , \tag{95}$$

$$K' = - N' L' [(\lambda_m^4) (\lambda_m^4 + Q_\infty)] + \lambda_m^8 (Q_\infty + \lambda_m^4) . \tag{96}$$

In (93), $im \, (q_1 \tan \tfrac{1}{2} q_1)$ must be evaluated in accordance with (84). The secular eqn. (92) has been solved in the second approximation for number of Q_∞ and the minimum value of T_∞ has been determined for both the cases. The results of calculations are summarized in Table 2.

TABLE 2

CRITICAL TAYLOR NUMBERS AND RELATED CONSTANTS FOR DIFFERENT VALUES OF Q_∞

		Second approximation	
Q_∞	R_∞	T_∞	Corresponding value of T_∞ as per Chandrasekhar
225	$-0 \cdot 4771$	$2 \cdot 9891 \times 10^5$	$2 \cdot 4122 \times 10^4$
2700	$-0 \cdot 7045 \times 10^{-2}$	$1 \cdot 2042 \times 10^7$	$1 \cdot 2184 \times 10^6$

The corresponding asymptotic behaviours are:

For $Q_\infty = 225\,,\ \overline{T} = 1329{\cdot}5\ Q\,,\ \left[\overline{T} = 107{\cdot}2\ Q \text{ (as per Chandrasekhar)}\right]$

and $$a \rightarrow \frac{15{\cdot}0}{\sqrt{Q}} \text{ as } Q \rightarrow \infty\,, \tag{97}$$

For $Q_\infty = 2700,\ \overline{T} = 4459{\cdot}9\ Q\,,\ \left[\overline{T} = 451{\cdot}27\ Q \text{ (as per Chandrasekhar)}\right]$

and $$a \rightarrow \frac{52{\cdot}0}{\sqrt{Q}} \text{ as } Q \rightarrow \infty\,. \tag{98}$$

It is observed that the critical Taylor numbers in the presence of an axial volume current superposed by an axial uniform magnetic field at which the instability sets-in, are increased.

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