# SUFFICIENT CONDITIONS FOR STABILITY OF COMPLETELY CONFINED FLUIDS IN PRESENCE OF MAGNETIC FIELD

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Effect of magnetic field on the electrically conducting incompressible fluid, completely confined in a smooth container of arbitrary shape, has been studied. The temperature gradient, concentration gradients and time independent magnetic field are assumed to act parallel to body force. Two Rayleigh numbers i.e. Modified Rayleigh number $MR$ and critical Rayleigh number $Rc$ have been obtained. It is found that instability occurs if $MR < Rc$.

## NOTATIONS

| | |
|---|---|
| $Amnp$, $Bmnp$ | Coefficient in Fourier series |
| $C$ | Complex space, concentration |
| $d$ | Contribution to body force per unit disturbance |
| $F$, $G$ | Function space |
| $K$ | Diffusivity |
| $k$ | Inverse of Prandtl number |
| $\nu$ | Kinematic viscosity |
| $\eta_H$ | Magnetic diffusivity |
| $\gamma$ | Number of solvents |
| $m, n, p$ | Mode numbers in Fourier series |
| $RH$ | Magnetic pressure number |
| $\beta$ | Gradient |
| $\alpha$ | Change of density per unit gradient |
| $r$ | Disturbance |
| $\sigma$ | Complex eigenvalue |
| $or$ | At origin |
| $\lambda$ | Complex eigenvalue |
| $o$ | Initial rest state |
| $\phi$ | Cartisian 2 $(\gamma+1)$, dimensional parametric space |
| $P_2$ | $\nu/\eta_H$ |

The rest notations used in the problem have got its usual meanings.

The stability of laminar flows has been studied by many researchers. Lord Rayleigh[1] has studied theoretically the laminar flows. Chandrasekhar[2] has studied the problem of stability of these flows in details. He extended his studies to include the effect of magnetic field on stability of fluids. D. Pnueli et al[3] have investigated the sufficient condition for stability of a fluid completely confined in a closed container of arbitrary shape. The concentration and temperature gradients act, on the fluid, parallel to body force $g$ that is $z$-axis.

In this paper we have generalised the stability condition[3] in presence of a time independent magnetic field $H_0$.

## BASIC EQUATION

Let $H_0 = (0,0\, H_z)$ be the uniform magnetic field acting on an electrically conducting incompressible fluid parallel to $z$-axis. $\mu_e$ the magnetic permeability is taken to be constant. The basic equations for temperature, concentration and M.H.D. flow are given by :

$$\nabla \cdot \bar{U} = 0 \tag{1}$$

$$\frac{D\bar{U}}{Dt} - \frac{\mu_e}{\rho} (\bar{H}\cdot\nabla)\, \bar{H} = -\nabla P_t - g\hat{k} + \nu\nabla^2 \bar{U} \tag{2}$$

where

$$P_t = \frac{P}{\rho} + \frac{\mu_e}{\rho} \frac{(\bar{H})^2}{2}$$

$$\left(\frac{D}{Dt} - \eta_H \nabla^2\right) \bar{H} = \nabla \times (\bar{U} \times \bar{H}) \tag{3}$$

$$\left(\frac{D}{Dt} - K_o \nabla^2\right) T = 0 \tag{4}$$

$$\left(\frac{D}{Dt} - K_j \nabla^2\right) \rho_j/\rho = 0\,, \quad j = 1, 2, 3 \ldots\ldots \gamma \tag{5}$$

A constant temperature gradient is set in the fluid parallel to $z$-axis i.e.

$$\nabla \overset{o}{T} = \hat{k} \frac{\partial \overset{o}{T}}{\partial z} = \beta_o \hat{k} \tag{6}$$

A density field is assumed to be linearly dependent on the temperature field :

$$\nabla \rho_0 = \alpha_0 \nabla \overset{o}{T} \tag{7}$$

A constant concentration gradient of some solvent[3] is now set on the fluid parallel to $z$-axis by imposing an appropriate concentration as a boundary condition. An additional change in the density field is assumed to be superimposed on the previous one i.e.

$$\nabla \rho = \nabla \rho_0 + \nabla \rho_1 = \alpha_o \nabla \overset{o}{T} + \alpha_1 \nabla \overset{o}{C_1} = k\,(\alpha_o \beta_o + \alpha_1 \beta_1) \tag{8}$$

This process is repeated till there is $\gamma$ concentration and temperature gradient, all parallel to $z$-axis and contributing to the density gradient. The fluid is initially at rest. The body force $g$ is also acting along $z$-axis.

The initial rest state of the fluid is given by :

$$\left.\begin{aligned} \overset{o}{q} &= 0 \\ \overset{o}{\rho_j}/\overset{o}{\rho} &= \overset{o}{\rho}_{jor}/\overset{o}{\rho}_{or} + \alpha_j \beta_j z \\ \overset{o}{T} &= \overset{o}{T}_{or} + \beta_o z \\ \overset{o}{\rho} &= \overset{o}{\rho}_{or} (1 + \sum_{j=0}^{\gamma} \alpha_j \beta_j z) \\ \frac{dP}{dz} &= - \overset{o}{\rho} g \end{aligned}\right\} \tag{9}$$

Since there is no motion, this initial state must persist. However, small disturbances may cause spontaneous flow. Cases in which any small disturbance decays in time are defined as stable, while all other cases are unstable. Sufficient conditions are sought for a case to be stable.

The 'principle of exchange of stabilities' is known to hold good for some cases.

## BASIC ANALYSIS

Let the initial state described by equation (9) be slightly perturbed. The perturbation in pressure temperature, magnetic field and density is denoted by $\delta p$, $r_o$, $h$, $\delta\rho$ respectively. Let $u,v,w$ denote the velocity in perturbed state. Equation of state for $\rho$ is substituted in momentum equation and the set is linearized.

The change in density $\delta\rho$ due to perturbation in temperature $r_o$ and $r_j$ in concentration is given by :

$$\delta\rho = -\alpha\rho\,(r_o + r_j) = -\alpha\rho_{or}\,[1 + \alpha_o (T - T_{or}) + \sum_{j=1}^{\gamma} \alpha_j (\rho_j/\rho - \rho_{jor}/\rho_{or})] \tag{10}$$

The set is rewritten in the non-dimensional form using following characteristic values[3]

$$\overset{*}{x} = \frac{x}{c} \quad \overset{*}{u} = \frac{u}{\nu c^{-1}} \quad \overset{*}{t} = \frac{t}{\nu^{-1} c^{2}} \quad \overset{*}{H_o} = \frac{H_o}{H} \quad \overset{*}{K_j} = \frac{K_j}{\nu}$$

$$\overset{*}{\alpha_j \beta_j} = \frac{\alpha_j \beta_j}{\nu^2 g^{-1} c^4} \quad \frac{\delta \overset{*}{P}}{\rho} = \frac{\delta P/\rho}{\nu^2 c^2} \quad \overset{*}{r_j} = r_j/\nu c K_j^{-1}$$

The disturbance is assumed to depend on the time in the form :

$$q(x,y,z,t) = e^{\sigma t} q(x,y,z), \; \sigma = R(\lambda) + i I(\lambda)$$

$$r_j(x,y,z,t) = e^{\sigma t} r_j(x,y,z), \; j = 0, 1,2 \ldots\ldots\ldots \gamma$$

We consider here the problem on the basis of Boussinesq's approximation[2]. By ignoring terms o second and higher orders in perturbation, equation (1) to (5) now becomes :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{11}$$

$$(\sigma - \nabla^2) \bar{U} = -\nabla P_t - \hat{k} \sum_{j=0}^{\gamma} d_j r_j - R_H (H_z \cdot \nabla) \bar{h} \tag{12}$$

$$(\sigma p_z - \nabla^2) \bar{h} = H_z \frac{\partial w}{\partial z} \tag{13}$$

$$(\sigma/k_j - \nabla^2) \gamma_j + \omega = 0, \qquad \text{where } j = 0 \text{ for temperature} \tag{14}$$

$$j = 1,2,3, \ldots\ldots \gamma \text{ for concentration}$$

$$P_t = \frac{\delta P}{\rho_o} + R_H \frac{H_z^2}{2}$$

with the boundary condition $u = v = w = r_j = 0$.

$d_j = (\alpha_j \beta_j g c^4)/K_j$ is the non-dimensional $j$ contribution to the modification of the body force per unit disturbance.

$r_j$ is the $j$ disturbance in the concentration (for temperature, $j = 0$).

$k_j = K_j/\nu$ is the inverse of $j$ Prandtl number.

$R_H = \frac{\mu_e H^2 \nu^2}{\rho c^2}$ is the non-dimensional Magnetic pressure number.

Boundary conditions for magnetic field depend upon the electrical properties of medium adjoining the fluid. Here we are considering the case of medium adjoining the fluid is a perfect conductor. Boundary conditions will be[2] :

$h_z = 0$ on the bounding surface. Since $\partial w/\partial z = 0$ on a rigid boundary, it follows from equation (11) that,

$\nabla^2 h_z = 0$ on the bounding surface.

Taking curl of eq. (12), the set of governing equations now becomes :

$$(\sigma - \nabla^2)(\partial u/\partial y - \partial v/\partial x) = 0 \tag{15}$$

$$(\sigma - \nabla^2)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \sum_{j=0}^{\gamma} d_j \frac{\partial r_j}{\partial x} + R_H H_z \frac{\partial h_z}{\partial x \, \partial z} \tag{16}$$

$$(\sigma - \nabla^2)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial y}\right) = \sum_{j=0}^{\gamma} d_j \frac{\partial r_j}{\partial y} + R_H H_z \frac{\partial h_z}{\partial y \, \partial z} \tag{17}$$

$$\nabla \cdot \bar{U} = 0 \tag{18}$$

$$(\sigma\, p_2 - \nabla^2)\, \bar{h} = H_z \frac{\partial w}{\partial z} \tag{19}$$

$$(\sigma/k_j - \nabla^2)\; r_j + w = 0,\; j = 0,1,2,\ldots\ldots.\gamma \tag{20}$$

## ANALYSIS

In (16), (17) & (20) coefficient $d_j$ and $k_j$ show the physical properties of the fluid which has 2 $(\gamma+1)$ parameters. A 2 $(\gamma+1)$ dimensional space $\phi$ is defined by D. Pnueli[3] et al. Following the same method of solution and placing the container inside the parallelopiped of sides $a$, $b$ and $c$. The system has now its point in parametric space $\phi$, its subregion in function space $F$ and its region in the complex space $C$. Let all the functions in $F$ which are defined inside the container and zero on its wall be identically same in the part of parallelopiped not occupied by the container. Now $F$ function space which contains all functions that : *inside the container*—they are the solutions of eigenvalue problem and are continuous with continuous derivatives. They are zero *on the container* and identically zero *outside the container*.

This definition does not change $F$, but increases the region of the definition of its functions to the whole parallelopiped.

Taking curl of (15), (16) and (17) we get,

$$(\sigma - \nabla^2)\; \nabla^2 w + \sum_{j=0}^{\gamma} d_j\, a^2\, r_j - R_H\, H_z\, D\, (\nabla^2\, \bar{h}) = 0 \tag{21}$$

where

$$a^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \text{ and } D = \partial/\partial z$$

Now the solution of eigenvalue problem (15)—(20) must satisfy (21). Consider a new function space $G$ which contains all the functions with the properties ; *on the boundaries of parallelopiped*—they are zero. *Inside the parallelopiped*—they are continuous with piecewise continuous derivative and solve eigenvalue problem.

$$(\sigma - \nabla^2)\; \nabla^2 f_w + \sum_{j=0}^{\gamma} d_j\, a^2 f_j + R_H\, H_z\, (\nabla^2\, \bar{h}) = 0 \tag{22}$$

$$(\sigma/k_j - \nabla^2) f_j + f_w = 0 \quad , \quad j = 0,\, 1,\, 2,\, 3\ldots\ldots..\gamma \tag{23}$$

$$(\sigma\, p_2 - \nabla^2)\; \bar{h} = H_z\, D\, f_w \tag{24}$$

Making use of eq. (24) in eq. (22), we get :

$$(\sigma - \nabla^2)\,(\sigma p_2 - \nabla^2)\; \nabla^2 f_w + \sum_{j=0}^{\gamma} d_j\, a^2 f_j\, (\sigma p_2 - \nabla^2) - Q\, D^2\, \nabla^2 f_w = 0 \tag{25}$$

$$(\sigma/k_j - \nabla^2)\, f_j + f_w = 0 \quad , \quad j = 0,1,2,\, \ldots\ldots\gamma \tag{26}$$

where

$$Q = R_H\, H_z^2$$

Now $d_j$ and $k_j$ are for the same point $\phi$ in parametric space as that of eq. (21) and eq. (22), (23) & (24). After comparing the definition of function space $F$ and $G$ we find that all functions in $F$ satisfy function $G$ while all functions in $G$ do not satisfy the requirements of space $F$. Hence function space $F$ can be obtained from function space $G$ by imposing additional requirement but that will decrease the function space. Hence it is found that the space $G$ contains space $F$ in the sense that

the $W$ functions are a subgroup of the $f_w$ functions,

the $r_j$ functions are a subgroup of the $f_j$ functions,

and $\sigma$ complex numbers of the $\lambda$ numbers.

This result can be described by the form[3]

$$F \trianglelefteq G \tag{27}$$

$$C_{\sigma r} \trianglelefteq C\lambda \tag{28}$$

It is now proposed to find conditions under which $C$ does not contain eigenvalues with $R(\lambda) > 0$. Equation (28) states that these are sufficient conditions for $R(\lambda) < 0$, and, therefore, sufficient conditions for stability.

Because $R(\lambda) = 0$ separate regions of $R(\lambda) > 0$ from those of $R(\lambda) < 0$, attention is now drawn to necessary conditions for pure imaginary eigenvalues.

Assume $f_w$ and $f_j$ with their $\lambda$ to be known. These functions may be expanded into a series of some complete set of normal modes[2].

$$f_w = \sum_{mnp} Amnp \sin[\pi m(x - a/2)/a] \sin[\pi n(y - b/2)/b]. \sin \pi p[(z - c/2)/c] \quad (29)$$

$$f_j = \sum_{mnp} B_j mnp \sin[\pi m(x - a/2)/a] \sin[\pi n(y - b/2)/b]. \sin[\pi p(z - c/2)/c] \quad (30)$$

Substituting the eq. (29) and (30) in eq. (25) & (26) we get,

$$Amnp(\lambda p_2 + \epsilon^2)(\lambda + \epsilon^2)\epsilon^2 + \eta^2 \sum_{j=0}^{\gamma} B_j mnp(\lambda p_2 + \epsilon^2) d_j + Q\, Amnp(\epsilon^2 - \eta^2)\epsilon^2 = 0 \quad (31)$$

$$B_j mnp(\lambda/k_j + \epsilon^2) + Amnp = 0 \qquad j = 0,1,2,3, \ldots\ldots\gamma \quad (32)$$

where

$$\epsilon^2 = \eta^2 + \pi^2 p^2/c^2 = (\pi m/a)^2 + (\pi n/b)^2 + (\pi p/c)^2 \quad (33)$$

Putting eq. (32) in (31) for $B_j mnp$, we get

$$\epsilon^2(\lambda/k_j + \epsilon^2)\left[(\lambda + \epsilon^2)(\lambda p_2 + \epsilon^2) + Q(\epsilon^2 - \eta^2)\right]$$

$$= \eta^2 \sum_{j=0}^{\gamma} d_j(\lambda p_2 + \epsilon^2) \quad (34)$$

Now $\lambda = R(\lambda) + i\,I(\lambda)$

Put

$$\left.\begin{aligned} x &= \frac{\eta^2}{\epsilon^2 - \eta^2} & R_1 &= \sum_{j=0}^{\gamma} \frac{d_j}{(\epsilon^2 - \eta^2)^2} \\ \lambda_1 &= \frac{\lambda}{\epsilon^2 - \eta^2} & & \\ p_1 &= \frac{1}{k_j} & Q_1 &= \frac{Q}{\epsilon^2 - \eta^2} \end{aligned}\right\} \quad (35)$$

Following Chandrasekhar[2] and equating real and imaginary parts when $R(\lambda) = 0$. We obtain

$$R_1 x = (1 + x)^3 - \{I(\lambda_1)\}^2(1 + x)(p_1 + p_2 + p_1 p_2) + Q_1(1 + x) \quad (36)$$

and

$$R_1 x/(1 + x) = (1 + x)^2(1 + p_1 + p_2) - p_1 p_2\{I(\lambda_1)\}^2 + p_1 Q_1 \quad (37)$$

Substitute for $R_1 x$ from (37) in eq. (36)

$$\{I(\lambda_1)\}^2 p_2^2 = \frac{p_2 - p_1}{1 + p_1} Q_1 - (1 + x)^2 \quad (38)$$

Put $X = I(\lambda)/\epsilon^2$, and writing the values of $Q_1$ $x$ and $I(\lambda_1)$ from (35), we get

$$X^2 p_2^2 = \frac{p_2 - p_1}{1 + p_1} Q \frac{(\epsilon^2 - \eta^2)}{\epsilon 4} - 1 \quad (39)$$

and the values of $X$ are the real solutions[3] of (39), which may have more than one solution. Let the solutions are denoted by $X_m$, $m$=0, 1, 2.........., where $X_0 = 0$ and $X_m$, $m \neq 0$, are the non-zero real solutions of (39). On putting the values of $Q_1$, $R_1$ and $x$ in (36) from eq. (35), we get

$$\frac{\epsilon^2}{2}\left[\epsilon^4 + Q(\epsilon^2 - \eta^2)\right] = \sum_{j=0}^{\gamma} d_j \frac{(1 + p_1 p_2 X^2)}{1 + X p_1^2} - \frac{p_2 X^2}{\eta^2 \epsilon^2} \quad (40)$$

Consider the right hand side of (40). Let this expression attain a certain value when one of these $X_m$ is substituted in it; in general it becomes a different number for each of the $X_m$. Let the largest number so obtained be denoted by $MR$ (Modified Rayleigh Number) ;

$$MR = Max \left[ \sum_{j=0}^{\gamma} \quad d_j \; \frac{(1 + p_1 \, p_2 \, X_m{}^2)}{1 + X_m{}^2 \, p_1{}^2} \quad - \quad \frac{p_2 \, X_m{}^2}{\eta^2 \, \epsilon^2} \right] \text{ for } m = 0, 1, 2 \ldots, \tag{41}$$

$$\text{When } X = X_0 = 0 \;\; MR = \sum_{j=0}^{\gamma} d_j = \text{Rayleigh number } R_c$$

hence $MR > R_c$

Chandrasekhar[2] has defined $MR$ with various value of $p_1$ and $p_2$. To avoid the complications at this stage. we have taken $p_2$ as a constant. But $MR$ does depend on various values of $p_2$ and $p_1$.

Let critical Rayleigh number $R_c$ attain the minimum value of the expression $\epsilon^2/\eta^2 \, [\, \epsilon^4 + Q \, ( \, \epsilon^2 - \eta^2 \, ) \, ]$. Where $\epsilon$ and $\eta$ are the continuous function of $(m, n, p, a, b, c)$.

$$R_c = Min \; [\, \epsilon^2/\eta^2 \, \{ \epsilon^4 + Q \, ( \, \epsilon^2 - \eta^2 \, ) \, \} \, ] \tag{42}$$

This value $R_c$ depends on various values of $Q$. As $Q$ increases $R_c$ increases. The same can be determined by the table given by Chandrasekhar[2].

It should be noted that eq. (40) is not satisfied, for pure imaginary eigenvalues, when $MR < R_c$ is the condition taken, therefore no such values exist or the imaginary coordinate axis in the complex plane is a forbidden zone to eigenvalues. Whereas $MR > R_c$ does not show the same result because $\epsilon^6/\eta^2$ can always be largely chosen to satisfy (40).

Analysing (33): the eigenvalues $\lambda$ are the solutions of a polynomial, and, therefore, they are continuous functions of its coefficients. For larger values of $p$ and moderate values of $m$ and $n$ (34) becomes

$$\lambda \approx - \epsilon^2 \tag{43}$$

This shows that there are always values of $\lambda$ in the left hand side of complex plane. However when the imaginary axis is a forbidden zone the $\lambda$ value cannot pass continuously into right hand side of the complex plane. Therefore, the condition

$$MR < R_c \tag{44}$$

is sufficient for stability.

## CONCLUSION

The two Rayleigh numbers (Modified & Critical) have been found out in presence of Magnetic field. The value of $R_c$ depends on the sides of parallelopiped and the strength of magnetic field. $R_c$ increases as the value of magnetic field increases. It is seen from eq. (41) that Modified Rayleigh number $MR$ does not depend upon sides of parallelopiped but it is dependent upon $p_2$ and hence the presence of magnetic field decreases the value of $MR$. Further it is noted that the presence of a magnetic field makes the condition of stability $MR < R_c$ satisfied for the unstable modes in the case when the magnetic field is not acting on the fluids.

The laminar flow, with a pressure gradient is identical with a flow of lubricating oil, in the narrow space in between journal and bearing, which is used in all moving parts of *aircrafts* and etc.

## REFERENCES

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