A NOTE ON WAVES GENERATED AT A LIQUID-SOLID INTERFACE-II (VISCOUS EFFECTS)

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The viscous effects of a liquid layer overlying a solid halfspace (assumed generalized thermoelastic) are examined.

This paper, being a continuation of a previous note of the same title, deals with the effect of viscosity on the interface waves propagated in a liquid layer overlying a generalized thermoelastic solid halfspace. This note extends the results of Harinath[1] and has applications to defence science and geophysical problems. The problem considered has more relevance to the physical situations encountered in reality than its elastic counterpart. In other words, generalized thermoelasticity fits into the realistic situation better than classical elasticity or coupled thermoelasticity. The details pertaining to the non-viscous effects of liquid layer overlying a perfectly elastic halfspace may be found in the treatise by Ewing, Jardetzky and Press[2]; those of coupled thermoelasticity in the treatise by Nowacki[3]; and details of wave propagation problems in generalized thermoelasticity in the concise paper by Harinath[4]. The stress-strain relations used for a viscous fluid may be found in Pipkin[5], with slight modifications, without any loss in generality. Most of the results obtained here are new.

BASIC EQUATIONS

Let us consider a homogeneous isotropic generalized thermoelastic halfspace with an overlying incompressible viscous liquid layer, both initially maintained at a constant reference temperature T_0. We set up a rectangular cartesian coordinate system (x, y, z) in the media such that the free surface of the liquid is chosen as the plane containing the x and y axes and the media is represented by $z \geqslant 0$ by taking the z-axis vertically downwards. If H is the thickness of the liquid layer, then $0 \leqslant z < H$ represents the liquid layer, $z = H$ is the interface and $z \geqslant H$ is the solid halfspace. (Any other convenient system of coordinates may be used. This, however, does not alter the frequency equation because of the tensorial behaviour of the stresses and strains).

Suppose the liquid (assumed to be incompressible and viscous in what follows) is of density ρ', of hydrostatic pressure p, and of sound velocity μ (i. e. the velocity of sound waves in the liquid is μ). For waves propagated in the x-direction with no components along the y-direction the displacement components $(U', 0, W')$ in the liquid are given by

$$U' = \frac{\partial \phi}{\partial x} = \phi'_x \, , \; W' = \frac{\partial \phi}{\partial z} = \phi'_z \tag{1}$$

where the potential function ϕ' satisfies the wave equation

$$\mu^2 (\phi'_{xx} + \phi'_{zz}) = \ddot{\phi}' \tag{2}$$

wherein a superimposed dot denotes differentiation with respect to the time variable $\tilde{t}$. Also the hydrostatic pressure p in the liquid may be expressed in terms of ϕ' as

$$p = - \rho' \dot{\phi}' .$$

Let the material of the solid halfspace $z \geqslant H$ be of density ρ, specific heat at constant strain s, coefficient of thermal conductivity k, ratio of the coefficient of linear thermal expansion to isothermal compressibility being γ, relaxation time factor t', coupling constant ϵ, isothermal longitudinal wave velocity α and shear

*Present address : Department of Mathematics, Bangalore University, BANGALORE-560001 wave velocity β and further, let T denote the temperature deviation from T_0 after a lapse of time $\tilde{t}$. Actually, $\epsilon = \gamma^2 T_0/\rho^2 s\alpha^2$ is of order 10^{-2} for common metals while the relaxation time factor t' is of order 10^{-14}. For waves propagated along the x-direction with no components along the y-direction the displacement components (U, O, W) may be chosen in terms of two potential functions ϕ, ψ as

$$U = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} = \phi_x - \psi_z, \qquad W = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} = \phi_z + \psi_x \tag{3}$$

where ϕ, ψ satisfy the partial differential equations

$$\begin{aligned} &\rho\alpha^2 (\phi_{xx} + \phi_{zz}) - \gamma (T + t' \ddot{T}) = \rho\ddot{\phi} \\ &\beta^2 (\psi_{xx} + \psi_{zz}) = \ddot{\psi} \\ &\rho s (\dot{T} + t' \ddot{T}) + \gamma T_0 (\dot{\phi}_{xx} + \dot{\phi}_{zz} + t' \ddot{\phi}_{xx} + t' \ddot{\phi}_{zz}) = k (T_{xx} \mp T_{zz}) \end{aligned} \tag{4}$$

In short, we are assuming all quantities to be independent of y, thereby reducing the problem to one of two dimensional plane strain. It would not be out of place to remark that in a generalized thermoelastic halfspace, the longitudinal wave velocity exceeds α, actually equal to $\alpha\sqrt{1+\epsilon t}$, where $t=1 - iwt'$. [see equation (7)]. In reality there exists no thermoelastic solid with longitudinal wave velocity equal to α. However, following Chadwick, we use α for the sake of notational convenience and we refer to it as the isothermal longitudinal wave velocity.

In terms of a frequency parameter w and a wave number δ, for waves propagated in the x-direction, we assume the potential functions ϕ', ϕ, ψ and the temperature deviation T to be given by

$$\left.\begin{aligned} \phi' &= \bar{\phi}' \exp i (\delta x - w\tilde{t}) \\ \phi &= \bar{\phi} \exp i (\delta x - w\tilde{t}) \\ \psi &= \bar{\psi} \exp i (\delta x - w\tilde{t}) \\ T &= \bar{T} \exp i (\delta x - w\tilde{t}) \end{aligned}\right\} \tag{5}$$

wherein through a suitable Fourier transform with respect to w the simple harmonic time dependence facto $\exp(-iw\tilde{t})$ could be converted into other time—dependence factors.

Substitutions of equations (5) in the earlier equations lead to

$$\left.\begin{aligned} &\mu^2(\bar{\phi}'_{xx} + \bar{\phi}'_{zz}) = - w^2 \bar{\phi} \\ &p = i\, w\rho' \bar{\phi}' \\ &\rho\alpha^2 (\bar{\phi}_{xx} + \bar{\phi}_{zz}) - \gamma\, t\, \bar{T} = - \rho\, w^2 \bar{\phi} \\ &\beta^2 (\bar{\psi}_{xx} + \bar{\psi}_{zz}) = - w^2\bar{\psi} \\ &iw\rho st\, \bar{T} + i\, w\gamma t\, T_0 (\bar{\phi}_{xx} + \bar{\phi}_{zz}) = - k (\bar{T}_{xx} + \bar{T}_{zz}) \end{aligned}\right\} \tag{6}$$

where
$$t = 1 - i\, wt'. \tag{7}$$

For small frequencies, it may be observed that t is almost equal to 1.

As in[4] equations (6) may be solved to yield

$$\left.\begin{aligned}
\phi' &= \left[A'e^{-\lambda_0 z} + B'e^{\lambda_0 z}\right] e^{i(\delta x - w\tilde{t})} \\
\phi &= \left[Ae^{-\lambda_1 z} + Be^{-\lambda_2 z}\right] e^{i(\delta x - w\tilde{t})} \\
\psi &= Ce^{-\lambda_3 z} e^{i(\delta x - w\tilde{t})} \\
\gamma t T &= \rho\left[Ae^{-\lambda_1 z}(w^2 - \alpha^2 q_1^2) + Be^{-\lambda_2 z}(w^2 - \alpha^2 q_2^2)\right] e^{i(\delta x - w\tilde{t})}
\end{aligned}\right\} \quad (8)$$

in conformity with the notations used in[1] given by

$$q_0^2 = w^2/\mu^2 \quad ; \quad q_3^2 = w^2/\beta^2 \quad (9)$$

q_1^2, q_2^2 are the roots of the quartic equation

$$k\alpha^2 q^4 - q^2 [kw^2 + iw\rho st\,\alpha^2 (1 + \epsilon t)] + iw^3 \rho st = 0 \quad (10)$$

$$\lambda_j^2 = \delta^2 - q_j^2 , \; Re\,(\lambda_j) \geqslant 0, \; j = 0, 1, 2, 3 \quad (11)$$

The unknowns A', B', A, B, C in equations (8) are to be determined by using boundary conditions and for obtaining finite solutions it is clear that ϕ, ψ, $T \to 0$ as $z \to \infty$.

The normal stress σ_{zz} and the shear stress σ_{xz} in the solid medium are given by the expressions

$$\left.\begin{aligned}
\sigma_{zz} &= \rho\alpha^2 (\phi_{xx} + \phi_{zz}) + 2\rho\beta^2 (\psi_{xz} - \phi_{xx}) - \gamma t T \\
\sigma_{xz} &= \rho\beta^2 (2\phi_{xz} + \psi_{xx} - \psi_{zz})
\end{aligned}\right\} \quad (12)$$

Equations (12) are derived from a generalization of Duhamel-Neumann law[4], using the last of equations (4) which represents the modification of Fourier's law of heat conduction to include second order time derivatives yielding only a finite speed for the propagation of heat unlike coupled thermoelasticity.

For the incompressible viscous liquid layer, we use the following stress-strain relations given in[5]

$$\sigma'_{ij} = p\delta_{ij} + 2v\, e'_{ij} \quad (13)$$

with self-explanatory notations where in v represents the viscosity coefficient. More clearly, for normal and shear components of stresses from (13), we obtain

$$\left.\begin{aligned}
\sigma'_{zz} &= iw\rho'\phi' + 2v\,\phi'_{zz} \\
\sigma'_{xz} &= 2v\,\phi'_{xz}
\end{aligned}\right\} \quad (14)$$

(The slight modifications from[5] will entail no loss in generality).

Boundary conditions

The precise boundary conditions are

$$\left.\begin{aligned}
&\text{(i)} & \sigma'_{zz} &= 0 & &\text{on } z = 0 \\
&\text{(ii)} & \sigma_{zz} &= \sigma'_{zz} & &\text{on } z = H \\
&\text{(iii)} & \sigma_{xz} &= \sigma'_{xz} & &\text{on } z = H \\
&\text{(iv)} & W &= W' & &\text{on } z = H \\
&\text{(v)} & T &= 0 & &\text{on } z = H
\end{aligned}\right\} \quad (15)$$

i.e. we assume that the free surface is stress-free and at the interface all continuity conditions are satisfied. At the interface, no condition on the tangential displacement can be imposed.

FREQUENCY EQUATION

The frequency equation is determined by the elimination of the unknowns A', B', A, B, C present in the solutions (8) by the use of the boundary conditions (15).

A calculation leads to the following five equations in the unknowns A', B', A, B, C

$$\left.\begin{aligned}
&(iw\rho' + 2v\lambda_0^2)(A' + B') = 0\\
&(iw\rho'+2v\lambda_0^2)\left[e^{-\lambda_0 H} A' + e^{\lambda_0 H} B'\right] - \rho\,(2\beta^2\delta^2 - w^2)\left[e^{-\lambda_1 H} A + e^{-\lambda_2 H} B\right] +\\
&2i\delta\lambda_3\,\rho\beta^2 e^{-\lambda_3 H} C = 0\\
&-2i\delta v\lambda_0\left[e^{-\lambda_0 H} A' - e^{\lambda_0 H} B'\right] + 2i\delta\rho\beta^2\left[\lambda_1 e^{-\lambda_1 H} A + \lambda_2 e^{-\lambda_2 H} B\right] +\\
&\rho\,(2\beta^2\delta^2 - w^2)\,e^{-\lambda_3 H} C = 0\\
&-\lambda_0 e^{-\lambda_0 H} A' + \lambda_0 e^{\lambda_0 H} B' + \lambda_1 e^{-\lambda_1 H} A + \lambda_2 e^{-\lambda_2 H} B - i\delta e^{-\lambda_3 H} C = 0\\
&\rho e^{-\lambda_1 H}(w^2 - \alpha^2 q_1^2) A + \rho e^{-\lambda_2 H}(w^2 - \alpha^2 q_2^2) B = 0
\end{aligned}\right\} \quad (16)$$

On further simplification, equations (16) are equivalent to the following system of equations

$$\begin{aligned}
&e^{\lambda_0 H} A' + e^{-\lambda_0 H} B' = 0\\
&(iw\rho' + 2v\lambda_0^2)(A' + B') + \rho(2\beta^2\delta^2 - w^2)(A + B) -\\
&\quad - 2i\delta\lambda_3\,\rho\beta^2 C = 0\\
&2i\delta\lambda_0 v\,(A' - B') + 2i\delta\rho\beta^2(\lambda_1 A + \lambda_2 B) + \rho(2\beta^2\delta^2 - w^2) C = 0\\
&-\lambda_0 A' + \lambda_0 B' - \lambda_1 A - \lambda_2 B + i\delta C = 0\\
&(w^2 - \alpha^2 q_1^2) A + (w^2 - \alpha^2 q_2^2) B = 0
\end{aligned} \quad (17)$$

Thus, the frequency equation is given by the following determinantal equation

$$\begin{vmatrix}
e^{\lambda_0 H} & e^{-\lambda_0 H} & 0 & 0 & 0\\
iw\rho' + 2v\lambda_0^2 & iw\rho' + 2v\lambda_0^2 & \rho(2\beta^2\delta^2 - w^2) & \rho(2\beta^2\delta^2 - w^2) & -2i\delta\rho\beta^2\lambda_3\\
2i\delta\lambda_0 v & -2i\delta\lambda_0 v & 2i\delta\lambda_1\rho\beta^2 & 2i\delta\lambda_2\rho\beta^2 & \rho(2\beta^2\delta^2 - w^2)\\
-\lambda_0 & \lambda_0 & -\lambda_1 & -\lambda_2 & i\delta\\
0 & 0 & w^2 - \alpha^2 q_1^2 & w^2 - \alpha^2 q_2^2 & 0
\end{vmatrix} = 0 \quad (18)$$

Alternatively, equation (18) is written

$$\begin{vmatrix}
iw\rho' + 2v\lambda_0^2 & 2\beta^2\delta^2 - w^2 & 2\beta^2\delta^2 - w^2 & -2i\delta\lambda_3\beta^2\\
-2i\delta\lambda_0 v & 2i\delta\lambda_1\beta^2 & 2i\delta\lambda_2\beta^2 & 2\beta^2\delta^2 - w^2\\
+\rho\lambda_0 & -\lambda_1 & -\lambda_2 & i\delta\\
0 & w^2 - \alpha^2 q_1^2 & w^2 - \alpha^2 q_2^2 & 0
\end{vmatrix} e^{2\lambda_0 H} =$$

$$= \begin{vmatrix}
iw\rho' + 2v\lambda_0^2 & 2\beta^2\delta^2 - w^2 & 2\beta^2\delta^2 - w^2 & -2i\delta\lambda_3\beta^2\\
2i\delta\lambda_0 v & 2i\delta\lambda_1\beta^2 & 2i\delta\lambda_2\beta^2 & 2\beta^2\delta^2 - w^2\\
-\rho\lambda_0 & -\lambda_1 & -\lambda_2 & i\delta\\
0 & w^2 - \alpha^2 q_1^2 & w^2 - \alpha^2 q_2^2 & 0
\end{vmatrix} \quad (19)$$

It may be noted that the fourth order determinants in equation (19) are similar and each may be obtained from the other by merely changing the sign of λ_0 throughout. Thus we have to evaluate the right hand side (RHS) of equation (19), to obtain the frequency equation. A calculation shows

$$\text{RHS (19)} = (\lambda_1 - \lambda_2) \left\{ \begin{array}{l} \alpha^2(\lambda_1 + \lambda_2)\,(2\beta^2\delta^2 - w^2)\,[2\delta^2 \lambda_0\, v - \rho\,(2\beta^2\delta^2 - w^2)] + \\ + [w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)] \cdot \\ \cdot\ [iw^3 \rho' + 4\lambda_0 \lambda_3 \beta^2\delta^2 v + 2vw^2 \lambda_0^2 - 4\rho\lambda_0\lambda_3\delta^2\beta^4] \end{array} \right\} \tag{20}$$

Using (20) in equation (19) yields the frequency equation given by

$$e^{2\lambda_0 H} \left[\begin{array}{l} w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)] \cdot [iw^3 \rho' + 2vw^2 \lambda_0^2 - 4\lambda_0\lambda_3 v \beta^2\delta^2 + 4\rho\lambda_0\lambda_3\delta^2\beta^4] \\ - \alpha^2 (\lambda_1 + \lambda_2)\,(2\beta^2\delta^2 - w^2)\,[\rho\,(2\beta^2\delta^2 - w^2) - 2\lambda_0\, v\delta^2] \end{array} \right] =$$

$$= \left[\begin{array}{l} [w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)] \cdot [iw^3 \rho' + 2vw^2\lambda_0^2 + 4\lambda_0\lambda_3\, v\beta^2 \delta^2 - 4\rho\lambda_0\lambda_3\delta^2\beta^4] \\ - \alpha^2 (\lambda_1 + \lambda_2)\,(2\beta^2\delta^2 - w^2)\,[\rho\,(2\beta^2\delta^2 - w^2) + 2\lambda_0\, v\delta^2] \end{array} \right] \tag{21}$$

A further simplification of (21) using

$$\tanh \lambda_0 H = \frac{e^{\lambda_0 H} - e^{-\lambda_0 H}}{e^{\lambda_0 H} + e^{-\lambda_0 H}} = \frac{e^{2\lambda_0 H} - 1}{e^{2\lambda_0 H} + 1} \tag{22}$$

leads to the following neater form of the frequency equation represented by equation (23)

$$\left[\begin{array}{l} [w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)]\,(iw^3 \rho' + 2vw^2\lambda_0^2) \\ - \rho\alpha^2 (\lambda_1 + \lambda_2)\,(2\beta^2 \delta^2 - w^2)^2 \end{array} \right] \tanh \lambda_0 H =$$

$$= 2\delta^2\lambda_0 \left[\begin{array}{l} 2\lambda_3\, \beta^2\,(v - \rho\beta^2)\,[w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)] - \\ - v\alpha^2 (\lambda_1 + \lambda_2)\,(2\beta^2\delta^2 - w^2) \end{array} \right] \tag{23}$$

In the absence of the viscosity coefficient v, the above frequency equation (23) reduces

$$\tanh \lambda_0 H = \frac{4\rho\lambda_0\, \lambda_3\, \delta^2\, \beta^4\, [w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)]}{\rho\alpha^2 (\lambda_1 + \lambda_2)\,(2\beta^2\, \delta^2 - w^2)^2 - iw^3\rho'\,[w^2 - \alpha^2 (\delta^2 + \lambda_1 \lambda_2)]} \tag{24}$$

Equation (24) is the frequency equation, when a non-viscous liquid layer overlies a generalized thermoelastic halfspace. Furthermore, equation (24) may be observed to be another form of equation (10) of reference [1]. This is because the changes between coupled thermoelasticity and generalized thermoelasticity do not explicitly appear due to the use of similar notations. The results obtained generalize the results of reference [1].

LIMITING CASES

The limiting cases of reference [1] may be repeated here. However, we discuss only two interesting cases viz., the case of large frequencies with fixed wave number and the case of an incompressible solid. For small frequencies, since t is almost equal to 1, the coupled thermoelasticity and the generalized thermoelasticity do not differ appreciably and hence this limiting case is only indicated.

(i) *Large frequencies*

Suppose we assume that δ is constant and w is real and large. Then we have the following approximations

$$q_1^2 = \frac{w^2}{\alpha^2} + \frac{iw\rho s\epsilon t^2}{k} - \frac{\rho^2 s^2 \epsilon^2 \alpha t^4}{k^2}$$

$$q_2^2 = \frac{iw\rho st}{k} + \frac{\rho^2 s^2 \epsilon^2 \alpha^2 t^4}{k^2}$$

$$\lambda_1 = \frac{iw}{\alpha} \left[1 - \frac{i\rho s\epsilon\alpha^2 t^2}{2kw} + \frac{\rho^2 s^2 \epsilon^2 \alpha^2 t^4}{2k^2\, w^2} - \frac{\delta^2\, \alpha^2}{2w^2} \right] \tag{25}$$

$$\lambda_2 = -(1-i)\sqrt{\frac{w\rho st}{2k}}\left[1 - \frac{i\rho s\epsilon^2\alpha^2 t^3}{2k^2 w} + \frac{ik\delta^2}{2\rho stw}\right]$$

$$\lambda_3 = \frac{iw}{\beta} - \frac{i\beta\delta^2}{2w}$$

$$\lambda_0 = \frac{iw}{\mu} - \frac{i\mu\delta^2}{2w}$$

taken from equations (12) and (13) of reference [4], where terms containing $1/w^2$, $1/w^3$, . . . are all omitted.

Substitution of (25) in (23) after a lengthy calculation yields the frequency equation as a complicated mathematical expression which does not appear to have any immediate practical use. Hence, for the sake of brevity, we omit this cumbersome equation and consider only a particular case of it given by

$$\left.\begin{aligned} &\left[w^2 - \alpha^2\delta^2 + \alpha(1+i)\sqrt{\frac{w^3\rho st}{2k}} - \alpha^3(1-i)\sqrt{\frac{w\rho^3 s^3\epsilon^2 t^5}{8k^3}}\right] \cdot \\ &\cdot\left[w^2 \tan\frac{wH}{\mu}\left(iw\rho' - \frac{2vw^2}{\mu^2}\right) - 4iw\beta . \frac{\delta^2}{\mu}\cdot(v - \rho\beta^2)\right] = \\ &= \alpha^2\left[\frac{iw}{\alpha} + \frac{\rho s\epsilon\alpha t^2}{2k} - (1-i)\sqrt{\frac{w\rho st}{2k}}\right](2\beta^2\delta^2 - w^2)\cdot \\ &\cdot\left[\rho\tan\frac{wH}{\mu}(2\beta^2\delta^2 - w^2) - 2vw\alpha^2\frac{\delta^2}{\mu}\right] \end{aligned}\right\} \qquad (26)$$

The terms containing even δ^2/w are neglected in comparison with w.

From equation (26) one can calculate the phase-velocity and the attenuation of the waves propagated in the x-direction. It is easy to observe that equation (26) yields an equation for $V = w/\delta$ wherein all the terms containing δ may be replaced by V and w. Solving this equation in which V is determined in terms of w we can calculate the phase-velocity of the propagated waves [equal to $1/\mathrm{Re}(1/V)$] and the attenuation in the x-direction [equal to $w.\mathrm{Im}\,(1/V)$]. Hence to obtain meaningful expressions, it follows that we have to necessarily choose δ complex, since w is real. Also various interesting cases of equation (26) may be discussed for $0 \leqslant H \leqslant |\pi\mu/2w|$ · (See reference [2] for details).

For still larger frequencies, equation (26) reduces to

$$\left[w + \alpha(1+i)\sqrt{\frac{\rho stw}{2k}}\right]\cdot\left((i\rho' - \frac{2vw}{\mu^2}\right) = \rho\alpha^2\left[\frac{iw}{\alpha} + \frac{\rho s\epsilon\alpha t^2}{2k} - (1-i)\sqrt{\frac{w\rho st}{2k}}\right] \qquad (27)$$

It is very interesting to note that equation (27) is independent of H, the depth of the liquid layer and yields an approximation for the viscosity coefficient v of the liquid. An easy computation leads to

$$v = \frac{\mu^2}{2w\left[w + \alpha\sqrt{\frac{\rho stw}{2k}}\right]}\left[\alpha(\rho\alpha - \rho')\sqrt{\frac{\rho stw}{2k}} - \frac{\rho^2 s\epsilon\alpha^2 t^2}{2k}\right] \qquad (28)$$

which on further approximation reduces to

$$v = \frac{\alpha\mu^2(\rho\alpha - \rho')}{2w\left[\alpha + \sqrt{\frac{2kw}{\rho st}}\right]} \qquad (29)$$

In the absence of thermal terms, equation (29) assumes the form

$$v = \frac{\mu^2}{2w} (\rho\alpha - \rho') \tag{30}$$

Thus the viscosity coefficient may be computed at high frequencies by the approximation formula 29) which is rather simple. The corresponding equation (30) generalizes the results[2]. However, equaions (29) and (30) together show that the viscosity becomes vanishingly small as the frequency increases.

An equally surprising result comes out of equation (26) when the liquid layer is assumed to be very thin. It may be verified that we obtain the very same equation (27) when H is very small from (26). Thus very high frequencies or very thin liquid layer appear to have the same effect on the frequency equation at large frequencies. Also, it may be observed that the viscosity slightly increases the velocity of the propagated waves and the attenuation.

(*ii*) *Incompressibility*

Suppose we assume that the solid halfspace $z \geqslant H$ is incompressible. Then the isothermal longitudinal wave velocity is infinite and in the limiting case the frequency equation (23) assumes the form

$$\tanh(\lambda_0 H) = \frac{2\delta^2 \lambda_0 \left[2\lambda_3 \beta^2 (v - \rho\beta^2)(\delta^2 + \lambda_1\lambda_2) + v(\lambda_1 + \lambda_2)(2\beta^2\delta^2 - w^2)\right]}{(\delta^2 + \lambda_1\lambda_2)(iw^3\rho' + 2vw^2\lambda_o^2) + \rho(\lambda_1 + \lambda_2)(2\beta^2\delta^2 - w^2)^2} \tag{31}$$

where $\lambda_1^2 = \delta^2 - \dfrac{iw\rho st(1+\epsilon t)}{k}\alpha$ and $\lambda_2 = \delta$ (since α is infinite)

If we now assume that the liquid layer is also very thin, then equation (31) yields an approximation for H which takes the following form on further simplification for small frequencies

$$H = 2\delta^2 \left| \frac{4\delta^2\beta^2 \cdot v - w^2 \cdot v - 2\rho\delta^2\beta^4}{\delta(iw^3\rho' + 2w^2\delta^2 v) + \rho(2\beta^2\delta^2 - w^2)^2} \right| \tag{32}$$

Equation (32) is independent of thermal terms. Thus for small frequencies, the classical elasticity theory may be suitably modified to yield the required results. This is the reason, why we neglected the limiting cases for small frequencies.

CONCLUSIONS

Most of the limiting cases i.e. equations (26) to (32), reveal the fact that in the thermoelastic case w and δ are always coupled with t. For small frequencies, there are no appreciable changes. But there exists dispersion in all the cases due to the coupling of w, δ, t. Furthermore, the compressibility of the solid does not enter into consideration for small frequencies[1], in conjunction with the above remark. The presence of the thermal terms causes dispersion of the propagated waves and the presence of the viscosity coefficient slightly increases the velocity and attenuation of the propagated waves. Moreover, at very high frequencies or for a very thin liquid layer, the thermal terms have no influence on the viscosity, considered at large frequencies. Approximations to the viscosity coefficient v and the depth of the liquid layer H are obtained in some interesting cases and all the results are new and worth mentioning.

The above results enable us to compute the intensity and the frequency of the shock waves generated in an underground explosion caused by nuclear or conventional devices which are of prime importance in defence science, underground explosions and geophysics.

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