A NOTE ON THE FLOW OF VISCOELASTIC FLUIDS THROUGH A RECTILINEAR PIPE HAVING ITS CROSS-SECTION AS A PARALLELOGRAM WITH PRESSURE GRADIENT AS ANY FUNCTION OF TIME

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(*Received* 19 *November* 1976; *revised* 22 *September* 1977)

In this paper unsteady flow of viscoelastic fluids through a rectilinear pipe of uniform cross-section has been discussed. The section of the pipe is a parallelogram. A few particular cases of results with pressure gradient as any function of time i.e. flow under an impulsive pressure gradient, flow under constant pressure gradient and flow under harmonically oscillating pressure gradient have been discussed in detail.

Ghosh[1] has discussed the flow of viscoelastic fluids through rectangular ducts with pressure gradient as any function of time. We have discussed the same problem taking cross-section of tube as a parallelogram. The problem of Ghosh becomes a particular case of the problem.

GOVERNING EQUATIONS

According to Oldroyd the rheological equations satisfied by viscoelastic fluids (Oldroyd, 1958) are

$$p_{ik} = -p\delta_{ik} + p'_{ik}$$

$$p'_{ik} + \lambda_1 \frac{D}{Dt} p'_{ik} + \mu_0 p'_{jj} e_{in} - \mu_1 (p'_{ij} e_{jk} + p'_{jk} e_{ij}) - \nu_1 p'_{ji} e_{ji} \delta_{ik} =$$

$$= 2\eta_0 \left[e_{ik} + \lambda_2 \frac{D}{Dt} e_{ik} - 2\mu_2 e_{ij} e_{jk} + \nu_2 e_{ji} e_{ji} \delta_{ik} \right]$$

with the equation of incompressibility

$$e_{ii} = 0$$

where

$$\frac{D}{Dt} b_{ik} = \frac{\partial}{\partial t} b_{ik} + v_j b_{ik,j} + w_{ij} b_{jk} + w_{kj} b_{ij}$$

$$e_{ik} = \frac{1}{2} (v_{k,i} + v_{i,k})$$

$$w_{ik} = \frac{1}{2} (v_{k,i} - v_{i,k})$$

and δ_{ik} is the Kronecker delta, e_{ik} is the rate of strain tensor, p_{ik} the stress tensor, λ_1 the relaxation time, λ_2 the retardation time, η_0 the coefficient of viscosity and μ_0, μ_1, μ_2, ν_1 and ν_2 are material constants, each being of the dimension of time.

For

$$\eta_0 > 0, \quad \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geqslant 0$$

$$\mu_0 = \nu_1 = \nu_2 = 0$$

the liquid will exhibit Weissenberg climbing effect when sheared at a uniform rate between rotating cylinders.

For

$$\eta_0 > 0, \quad \lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0$$

$$\mu_0 = \nu_1 = \nu_2 = 0$$

the liquid will behave as ordinary viscous liquid.

EQUATION OF MOTION WITH INITIAL AND BOUNDARY CONDITIONS

The equations of motion in the absence of extraneous forces are

$$\rho \left[\frac{\partial v_i}{\partial t} + v_{i,j}\, v_j \right] = - p_j + p'_{ij,j} \tag{2}$$

where ρ is the density and p is the pressure.

We now consider a slow shearing motion through a rectilinear duct having its cross-section as a parallelogram with z-axis along the axis of duct. Initially liquid is at rest.

We assume $u = v = 0, \quad w = w\,(x, y, t)$.

The stress-strain relations for the present problem are as follows :

$$\left.\begin{aligned} \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p'_{zx} &= \eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \frac{\partial w}{\partial x} \\ \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) p'_{zy} &= \eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \frac{\partial w}{\partial y} \\ p'_{zz} = p'_{xx} = p'_{yy} &= p'_{xy} = 0 \end{aligned}\right\} \tag{3}$$

The equation of motion will thus be

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial w}{\partial t} = - \frac{1}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial z} + \nu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

where

$$\nu = \frac{\eta_0}{\rho}$$

Taking $-\dfrac{1}{\rho} \dfrac{\partial p}{\partial z}$ = any function of time $= f(t)$. Equation of motion becomes

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial w}{\partial t} = \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) f(t) + \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{4}$$

The boundary conditions are

(i) $w = 0$ at $y = mx + c, \quad y = mx, \quad my = - x + d, \; - my = - x$

(ii) $w = 0$ at $t \leqslant 0$

SOLUTION

Let us apply the linear transformation

$$\xi = y - mx \text{ and } \eta = my + x$$

Then the equation (4) becomes

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial w}{\partial t} = \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) f(t) + \nu\,(1 + m^2) \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \right) \tag{5}$$

and boundary conditions become

$$\left.\begin{aligned} &\text{(i) } w = 0 \quad \text{at } \xi = 0,\ c;\ \eta = 0,\ d \\ &\text{(ii) } w = 0 \quad \text{at } t \leqslant 0 \end{aligned}\right\} \tag{6}$$

To solve the equation (5) subject to the boundary conditions (6) we choose

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} A_{p,q}(t) \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d} \tag{7}$$

Since initially liquid is at rest, therefore

$$A_{p,q}(t) = 0 \quad \text{at } t \leqslant 0 \tag{8}$$

Then equation (5) gives

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} A_{p,q} = S_{p,q} \left[f(t) + \lambda_1 f'(t) \right] - \nu\gamma_{p,q} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) A_{p,q} \tag{9}$$

where

$$S_{p,q} = \frac{16}{pq\pi^2} \sin^2 \frac{p\pi}{2} \cdot \sin^2 \frac{q\pi}{2}$$

and

$$\gamma_{p,q} = (1 + m^2)\, \pi^2 \left(\frac{p^2}{c^2} + \frac{q^2}{d^2} \right)$$

Performing Laplace transform on (9) we get

$$\bar{A}_{p,q}(s) \left[\lambda_1 s^2 + (1 + \nu\lambda_2 \gamma_{p,q}) s + \nu \gamma_{p,q} \right] = S_{p,q} \left[\bar{f}(s) + \lambda_1 s \bar{f}(s) \right]$$

i.e.

$$\bar{A}_{p,q}(s) = \frac{S_{p,q} \{(1 + \lambda_1 s) \bar{f}(s)\}}{\lambda_1 s^2 + (1 + \gamma\lambda_2 \gamma_{p,q}) s + \nu\gamma_{p,q}}$$

$$= \frac{S_{p,q} (1 + \lambda_1 s) \bar{f}(s)}{\lambda_1 (\alpha - \beta)} \left[\frac{1}{s - \alpha} - \frac{1}{s - \beta} \right] \tag{10}$$

α, β are the roots of the equation

$$\lambda_1 s^2 + (1 + \nu\lambda_2 \gamma_{p,q}) s + \nu\gamma_{p,q} = 0$$

and

$$\alpha, \beta = \frac{-(1 + \nu\lambda_2 \gamma_{p,q}) \pm \sqrt{(1 + \nu\lambda_2 \gamma_{p,q})^2 - 4\lambda_1 \gamma_{p,q} \nu}}{2\lambda_1}$$

Applying convolution theorem on (10), we get

$$A_{p,q}(t) + \frac{S_{p,q}}{\lambda_1 (\alpha - \beta)} \left[\int_0^t e^{\lambda\alpha} \left\{ f(t - \lambda) + \lambda_1 f'(t - \lambda) \right\} d\lambda - \int_0^t e^{\lambda\beta} \left\{ f(t - \lambda) + \lambda_1 f'(t - \lambda) \right\} d\lambda \right] . \tag{11}$$

Hence

$$w(\xi,\eta,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{S_{p,q}}{\lambda_1(\alpha-\beta)} \left[\int_0^t e^{\lambda\alpha} \left\{ f(t-\lambda) + \lambda_1 f'(t-\lambda) \right\} d\lambda \right.$$

$$\left. - \int_0^t e^{\lambda\beta} \left\{ f(t-\lambda) + \lambda_1 f'(t-\lambda) \right\} d\lambda \right] \sin\frac{p\pi\xi}{c} \sin\frac{q\pi\eta}{d} \qquad (12)$$

If we transform ξ, η into x and y and put $m = 0$ then evidently the result coincides with that obtained by Ghosh[1].

At the points for which

$$\xi = \frac{c}{2p}, \text{ where } p = 1, 2, 3, \ldots\ldots\ldots$$

the velocity distribution is given by

$$w_c = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{S_{p,q}}{\lambda_1(\alpha-\beta)} \left[\int_0^t e^{\lambda\alpha} \left\{ f(t-\lambda) + \lambda_1 f'(t-\lambda) \right\} d\lambda \right.$$

$$\left. - \int_0^t e^{\lambda\beta} \left\{ f(t-\lambda) + \lambda_1 f'(t-\lambda) \right\} d\lambda \right] \sin\frac{q\pi\eta}{d}$$

$$\therefore \frac{w}{w_c} = \sum_{p=1}^{\infty} \sin\frac{p\pi\xi}{c} = \frac{1}{2}\cot\frac{\pi\xi}{2c}.$$

Graph of velocity profile is drawn in Fig. 1 with the help of following table :—

ξ	0	$\pm\frac{c}{3}$	$\pm\frac{c}{2}$	$\pm\frac{2c}{3}$	$\pm c$	$\pm\frac{4c}{3}$	$\pm\frac{3c}{2}$	$\pm\frac{5c}{3}$	$\pm 2c$
$\frac{w}{w_c}$	$\pm\infty$	$\pm$ 0·866025	$\pm$ 0·5	$\pm$ 0·28867	0	$\mp$ 0·28867	$\mp$ 0·5	$\mp$ 0·866025	$\mp\infty$

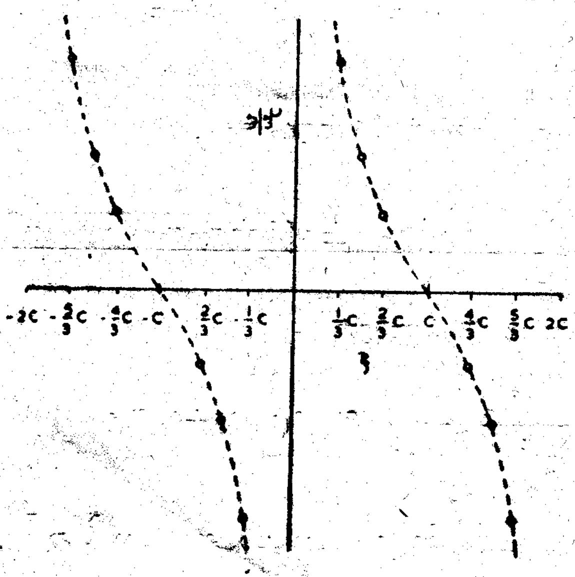

Fig. 1—Variation of $\frac{w}{w_0}$ and ξ.

Case I—Solution for Impulsive Pressure Gradient

Here $f(t) = A\,\delta(t)$

where

$$\int_{-\infty}^{\infty} f(\xi)\,\delta(\xi - c) = f(c)$$

Then from (12) we get

$$w(\xi, \eta, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{A S_{p,q}}{\lambda_1(\alpha - \beta)} \left[e^{\alpha t} - e^{\beta t} - \lambda_1 \alpha e^{\alpha t} + \beta \lambda_1 e^{\beta t} \right] \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d} .$$

If we transform ξ and η into x and y and put $m = 0$ the result coincides with that obtained by Ghosh[1].

Further solving for $\lambda_1 = 1 = \lambda_2$ we get

$$\alpha = -\nu\gamma_{p,q}, \quad \beta = -1$$

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{A S_{p,q}}{1 - \nu\gamma_{p,q}} \left(1 + \nu\gamma_{p,q}\right) \left(t - 1\right) e^{-t} \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d} .$$

neglecting second and higher powers of ν

at $t = 0$, velocity distribution is given by

$$w_0 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{A S_{p,q}}{1 - \nu\gamma_{p,q}} \left(1 + \nu\gamma_{p,q}\right) \left(-1\right) \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d} .$$

$$\frac{w}{w_0} = (1 - t)\, e^{-t}$$

Hence characteristic graph of velocity profile is given as under (Fig. 2).

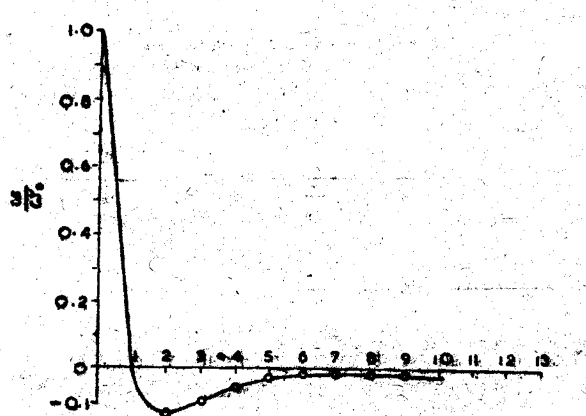

Fig. 2—Variation of $\frac{w}{w_0}$ and t.

t	0	1	2	3	4	5	6	7	8	9	10
$\frac{w}{w_0}$	1	0	−0·13534	−0·09958	−0·05496	−0·02696	−0·01240	−0·00547	−0·00231	−0·000987	−0·00041

Again when $\lambda_1 = 1,\ \lambda_2 = 2$

$$\alpha = \nu^2 \gamma_{p,q} - \nu\gamma_{p,q}\ ,\ \ \beta = -1 - \nu\gamma_{p,q} - \nu^2 {\gamma_{p,q}}^2$$

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{AS_{p,q}}{\sqrt{1 + 4\nu^2 {\gamma_{p,q}}^2}} \left[-1 + (2 + \nu\gamma_{p,q})\, t \right],$$

neglecting second and higher powers of ν

$\therefore$ at $t = 0$, velocity distribution is given by

$$w_0 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{AS_{p,q}}{\sqrt{1 + 4\nu^2 {\gamma_{p,q}}^2}} (-1)$$

$$\therefore \frac{w}{w_0} = 1 - (2 + \nu\gamma_{p,q})\, t$$

$$\therefore \left(\frac{w}{w_0} - 1 \right) \propto t$$

Characteristic graph is drawn in Fig. 3.

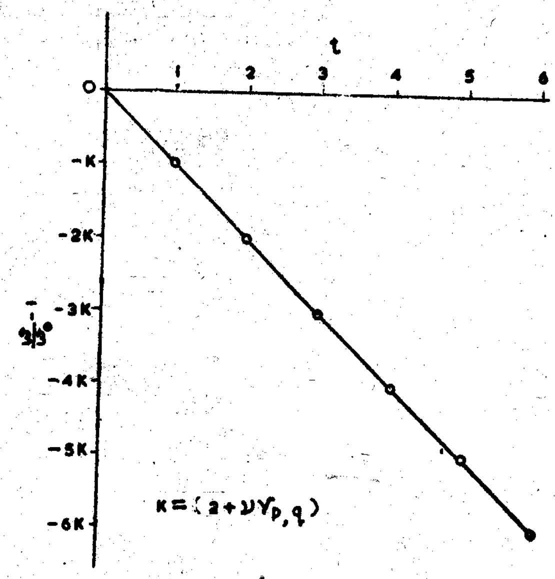

Fig. 3—Variation of $\left(\frac{w}{w_0} - 1 \right)$

For an ordinary viscous flow,

$$\lambda_1 \longrightarrow 0,\ \lambda_2 \longrightarrow 0$$

$$\alpha = -\nu\gamma_{p,q}\ ,\ \ \beta = -\infty$$

$$w(\xi, \eta, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} AS_{p,q}\, e^{-t\nu\gamma_{p,q}} \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d}$$

$$= \frac{16A}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)}\, e^{-t\nu \left[\frac{(2p+1)^2 \pi^2}{4c^2} + \frac{(2q+1)^2 \pi^2}{4d^2} \right]}$$

$$\times \cos \frac{(2p+1)\pi}{2c} \cos \frac{(2q+1)\pi}{2d} \qquad (13)$$

Equation (13) agrees with the result obtained by Ghosh[1] when we transform ξ and η into x and y and put $m=0$. Again if we put $A=1$ this expression agrees with that given by Fan and Chao[3].

Case II—Flow Under Constant Pressure Gradient

Here $f(t) = K_0 H(t)$

where $H(t)$ is Heaviside unit step function. Substituting it in (12) we get

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{S_{p,q} K_0}{\lambda_1(\alpha - \beta)} \left[\frac{e^{\alpha t}}{\alpha} - \frac{e^{\beta t}}{\beta} + \frac{\alpha - \beta}{\alpha\beta} + \lambda_1 \left(e^{\alpha t} - e^{\beta t} \right) \right] \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d}.$$

If we transform ξ, η into x and y and put $m = 0$, the result coincides with that obtained by Ghosh[1].

When $\lambda_1 = 1 = \lambda_2$, we have

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{S_{p,q} K_0}{(\nu\gamma_{p,q} - 1)} \quad t,$$

neglecting second and higher powers of ν

at $t = 1$, velocity distribution is given by

$$w_1 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{S_{p,q} K_0}{(\nu\gamma_{p,q} - 1)}$$

$$\therefore \frac{w}{w_1} = t$$

Hence graph of velocity profile is drawn in Fig. (4). Again when

$$\lambda = 1, \quad \lambda_2 = 2$$

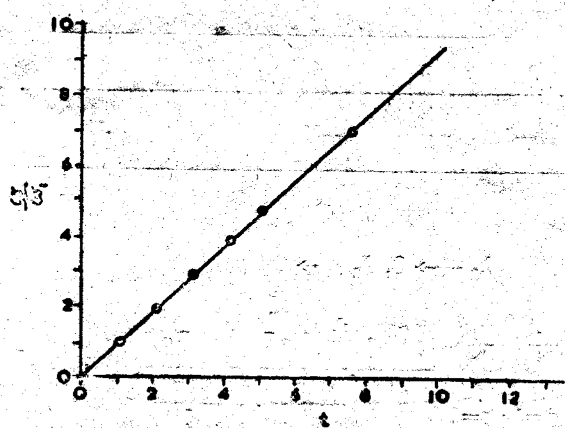

Fig. 4—Variation of $\frac{w}{w_1}$ and t.

$$\alpha = \nu^2 \gamma_{p,q}^2 - \nu\gamma_{p,q}, \; \beta = -1 - \nu\gamma_{p,q} - \nu^2 \gamma_{p,q}^2$$

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{K_0 S_{p,q}}{(1 - \nu\gamma_{p,q})} \cdot \nu\gamma_{p,q} (3t - 1).$$

At t=0, velocity distribution is given by

$$w_0 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{K_0 S_{p,q}}{1 - \nu\gamma_{p,q}} \quad \nu\gamma_{p,q} (-1)$$

$$\therefore \frac{w}{w_0} = (1 - 3t)$$

Graph of velocity profile is drawn as follows (Fig. 5).

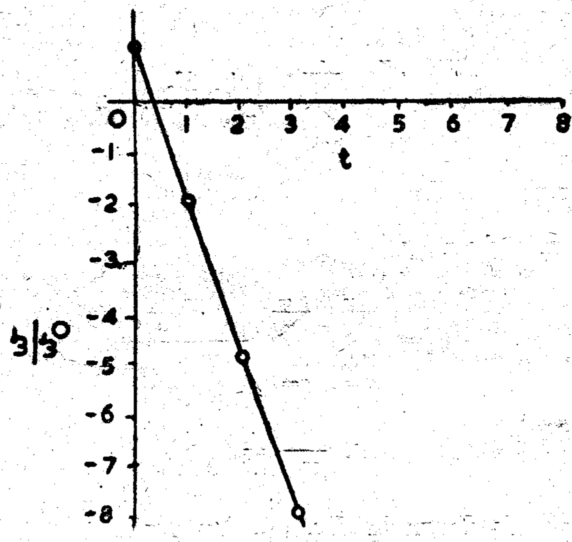

Fig. 5—Variation of $\frac{w}{w_0}$ and t.

t	0	1	2	3	4	5	6	7	8	9
$\frac{w}{w_0}$	1	—2	—5	—8	—11	—14	—17	—20	—23	—26

For an ordinary viscous flow

$$\lambda_1 \longrightarrow 0, \ \lambda_2 \longrightarrow 0$$

$$\therefore w(\xi, \eta, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{16 K_0 \sin^2 \frac{p\pi}{2} \ \sin^2 \frac{q\pi}{2}}{pq\pi^2 \nu \left[\frac{p^2\pi^2}{c^2} + \frac{q^2\pi^2}{d^2} \right]} \left[1 - e^{-\nu t \left(\frac{p^2\pi^2}{c^2} + \frac{q^2\pi^2}{d^2} \right)} \right] \times$$

$$\times \sin \frac{p\pi\xi}{c} \ \sin \frac{q\pi\eta}{d} \qquad (14)$$

Equation (14) agrees with that obtained by Ghosh[1] when we transform ξ and η into x and y and put $m = 0$ and further if we replace p by $(2p'+1)$ where p'=0, 1, 2, 3................. and q by $(2q'+1)$ where q' = 0, 1, 2, 3, 4.................. the result coincides with that obtained by Fan and Chao[3].

Case III—Flow Under Harmonically Oscillating Pressure Gradient

Here $f(t) = K \cos \omega t$.

Hence equation (12) gives

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{K S_{p,q}}{\lambda_1 (\alpha - \beta)} \left[\int_0^t e^{\lambda\alpha} \left\{ \cos \omega (t - \lambda) + \lambda_1 \omega \sin \omega (t - \lambda) \right\} d\lambda \right.$$

$$\left. - \int_0^t e^{\lambda\beta} \left\{ \cos \omega (t - \lambda) + \lambda_1 \omega \sin \omega (t - \lambda) \right\} d\lambda \right] \sin \frac{p\pi\xi}{c} \sin \frac{q\pi\eta}{d}$$

$$= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{KS_{p,q}}{\lambda_1 (\alpha - \beta)(\alpha^2 + \omega^2)} \Big[\Big\{ - \alpha \cos \omega t + \omega \sin \omega t + \alpha e^{\alpha t} +$$

$$+ \lambda_1 \omega (- \alpha \sin \omega t - \omega \cos \omega t + \omega e^{\alpha t}) \Big\} - \Big\{ - \beta \cos \omega t +$$

$$+ \omega \sin \omega t + \beta e^{\beta t} + \lambda_1 \omega (- \beta \sin \omega t - \omega \cos \omega t + \omega e^{\beta t}) \Big\} \Big] \sin \frac{p \pi \xi}{c} \sin \frac{\eta}{d}$$

When $\lambda_1 = 1 = \lambda_2$, we have

$$\alpha = - \nu \gamma_{p,q}, \quad \beta = -1$$

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{KS_{p,q}}{(1 - \nu \gamma_{p,q})(\nu^2 \gamma_{p,q} + \omega^2)} \; (-t)$$

neglecting higher powers of ν and ω

at $t = 1$, velocity distribution is given by

$$w_1 = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{KS_{p,q}}{(1 - \nu \gamma_{p,q})(\nu^2 \gamma_{p,q}^2 + \omega^2)} \; (-1)$$

$$\frac{w}{w_1} = t$$

Hence characteristic graph is drawn in Fig. 6. Similarly we can discuss the case when $\lambda_1 = 1$, $\lambda_2 = 2$.

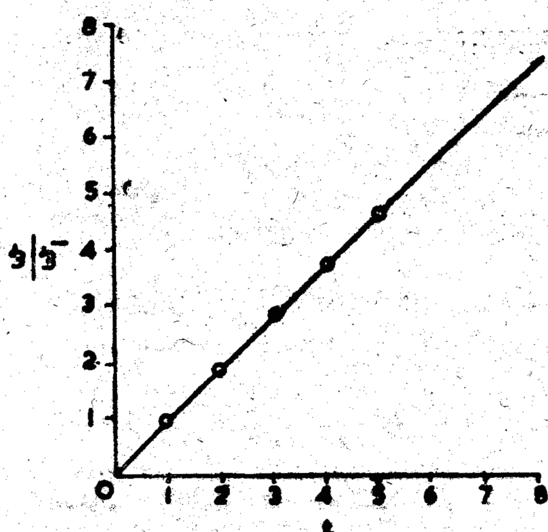

Fig. 6—Variation of $\frac{w}{w_1}$ and t.

For ordinary viscons flow,

$$\lambda_1 \longrightarrow 0, \quad \lambda_2 \longrightarrow 0$$

$$w = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{K S_{p,q}}{\omega^2 + \nu^2 \gamma_{p,q}^2} \left[\nu \gamma_{p,q} \cos \omega t + \omega \sin \omega t - \nu \gamma_{p,q} e^{-\nu t \gamma_{p,q}} \right] \sin \frac{p \pi \xi}{c} \sin \frac{q \pi \eta}{d} \quad (15)$$

where

$$S_{p,q} = \frac{16}{pq\pi^2} \sin^2 \frac{p\pi}{2} \sin^2 \frac{q\pi}{2}$$

$$\gamma_{p,q} = \left(1 + m^2 \right) \pi^2 \left(\frac{p^2}{c^2} + \frac{q^2}{d^2} \right)$$

Equation (15) also reduces to the result obtained by Ghosh[1] when we put $m = 0$ after transforming ξ and η into x and y. This can also be made to satisfy Fan and Chao[3]'s result after suitable adjustment.

DISCUSSION

Beyond the discussion given by Ghosh[1] we have obtained the approximate values of w for the different values of λ_1 and λ_2 and have drawn graphs showing the characteristics of viscoelastic flows. In Fig. 1 we see that at the points for which $\xi = \frac{c}{2p}$, $p = 1, 2, 3, 4, \ldots\ldots$, the velocity profile graph is a cotangent graph. Other approximate graphs are straight lines. By making $\lambda_1 \longrightarrow 0$, $\lambda_2 \longrightarrow 0$ we have obtained velocity distribution for ordinary viscous flow.

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