VIBRATIONS IN THICK HOLLOW ELASTIC SPHERES

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The problem of vibrations produced in a thick hollow sphere by the application of internal and external pressures which are functions of time is solved with the help of integral transform technique. A few special cases are illustrated numerically.

Sneddon[1] solved the problem of vibrations produced in a thick hollow elastic sphere by the application of only internal pressure, which is a periodic function of time having period $2\pi/w$.

This paper, deals with the more general problem of vibrations produced in a thick hollow elastic sphere by the application of internal as well as external pressures which are the general functions of time. Some of the special cases have been illustrated numerically.

STATEMENT OF THE PROBLEM

Consider a thick, isotropic, homogeneous hollow sphere of radii a and b deformed by the internal and external pressures $f_1(t)$ and $f_2(t)$ respectively. The radial component u, at radius r, and time t are then determined by the differential equation[1]

$$\frac{\partial^2}{\partial r^2} u + \frac{2}{r}\frac{\partial}{\partial r} u - \frac{2}{r^2} u = \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2} u \; ; \quad a < r < b, \; t > 0, \tag{1}$$

where the quantity α is defined in terms of the density ρ of the sphere and Lame's elastic constants λ and η by the relation

$$\alpha^2 = \frac{\lambda + 2\eta}{\rho} .$$

The radial component of the stress is given by

$$\sigma_r = (\lambda + 2\eta) \frac{\partial}{\partial r} u + 2\lambda \frac{u}{r}$$

so that the boundary conditions are

$$\left| (\lambda + 2\eta) \frac{\partial}{\partial r} u + 2\lambda \frac{u}{r} \right|_{r=a} = f_1(t) \; ; \quad t > 0 \tag{2}$$

$$\left| (\lambda + 2\eta) \frac{\partial}{\partial r} u + 2\lambda \frac{u}{r} \right|_{r=b} = f_2(t) \; ; \quad t > 0 . \tag{3}$$

The initial conditions are

$$u(r, 0) = u_0(r) \; ; \quad \frac{\partial}{\partial t} u(r, t)\Big|_{t=0} = u_0'(r) . \tag{4}$$

DEFINITION AND PROPERTIES OF AN INTEGRAL TRANSFORM

Following the procedure given by Marchi and Zgrablich[2] we define the integral transform $U^p(n)$ of the function $u(r)$ by the equation

$$U^p(n) = \int_a^b r^2 M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r)\, u(r)\, dr \tag{5}$$

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where

$$M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) = r^{-1/2} S_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r)$$
$$= r^{-1/2} [J_p(\mu_n r) \{ Y_p(\alpha_1, \alpha_2, \mu_n a) + Y_p(\beta_1, \beta_2, \mu_n b) \} -$$
$$- Y_p(\mu_n r) \{ J_p(\alpha_1, \alpha_2, \mu_n a) + J_p(\beta_1, \beta_2, \mu_n b) \}] \qquad (6)$$

and μ_n are the positive roots of the frequency equation

$$J_p(\alpha_1, \alpha_2, \mu_n a) Y_p(\beta_1, \beta_2, \mu_n b) - J_p(\beta_1, \beta_2, \mu_n b) Y_p(\alpha_1, \alpha_2, \mu_n a) = 0 \qquad (7)$$

in which

$$\left.\begin{aligned} J_p(s_1, s_2, \mu r) &= \left(s_1 - \frac{s_2}{2r} \right) J_p(\mu r) + \mu s_2 J'_p(\mu r) \\ Y_p(s_1, s_2, \mu r) &= \left(s_1 - \frac{s_2}{2r} \right) Y_p(\mu r) + \mu s_2 Y'_p(\mu r) \end{aligned}\right\} \qquad (8)$$

where $J_p(\mu x)$ and $Y_p(\mu x)$ are Bessel functions of first and second kind respectively.

The inversion of (5) is

$$u(r) = \sum_n \frac{1}{C_n} U^p(n) M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) \qquad (9)$$

where

$$C_n = \int_a^b r^2 \left[M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) \right]^2 dr = \left| \frac{r^2}{2} \left\{ S_p^2(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r) - \right.\right.$$
$$\left.\left. - T_{p-1}(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r) T_{p+1}(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r) \right\} \right|_a^b \qquad (10)$$

in which

$$T_{p\pm1}(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r) = J_{p\pm1}(\mu_n r) [Y_p(\alpha_1, \alpha_2, \mu_n a) + Y_p(\beta_1, \beta_2, \mu_n b)] -$$
$$- Y_{p\pm1}(\mu_n r) [J_p(\alpha_1, \alpha_2, \mu_n a) + J_p(\beta_1, \beta_2, \mu_n b)]$$

The operational property of the transform (5) is

$$\int_a^b r^2 \left[\frac{d^2}{dr^2} u + \frac{2}{r} \frac{d}{dr} u - \frac{p^2 - 1/4}{r^2} u \right] M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) \, dr$$
$$= \frac{b^2}{\beta_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b) \left[\beta_1 u + \beta_2 \frac{d}{dr} u \right]_{r=b}$$
$$- \frac{a^2}{\alpha_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a) \left[\alpha_1 u + \alpha_2 \frac{d}{dr} u \right]_{r=a} - \mu_n^2 U^p(n) \qquad (11)$$

Using the well-known results[3], it can be easily shown, for $p = 3/2$ and large n, that

$$\mu_n \approx \frac{n\pi}{b-a}, \qquad (12) \qquad M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) = O\left(\frac{1}{\mu_n}\right), \qquad (13)$$

and

$$C_n = O\left(\frac{1}{\mu_n^2}\right) \qquad (14) \qquad S_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n r) = O\left(\frac{1}{\mu_n}\right) \qquad (15)$$

SOLUTION

Applying transform with respect to r as defined in (5) alongwith its operational property (11) to equations (1) and (4), we get

$$\frac{b^2}{\beta_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b) f_2(t) - \frac{a^2}{\alpha_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a) f_1(t) - $$

$$- \mu_n^2 U^p(n, t) = \frac{1}{\alpha^2} \frac{d^2}{dt^2} U^p(n, t) \tag{16}$$

and

$$U^p(n, 0) = U_0^p(n) \; ; \quad \frac{d}{dt} U^p(n, t) \Big|_{t=0} = U_0'^p(n) \tag{17}$$

Applying Laplace transform with respect to t, defined by

$$\overline{U^p}(n, q) = \int_0^\infty U^p(n, t) \exp(-qt) \, dt$$

to equation (16) and using (17), we have

$$\overline{U^p}(n, q) = \left[\frac{b^2}{\beta_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b) \bar{f_2}(q) - \frac{a^2}{\alpha_2} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a) \right.$$

$$\left. \bar{f_1}(q) + q U_0^p(n) + U_0'^p(n) \right] \Big/ (q^2 + \mu_n^2 \alpha^2) . \tag{18}$$

Applying inverse Laplace transform and its convolution property to equation (18), we get

$$U^p(n, t) = \frac{b^2 M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b)}{\beta_2 \mu_n \alpha} \int_0^t \sin \mu_n \alpha (t - \xi) f_2(\xi) \, d\xi - $$

$$- \frac{a^2 M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a)}{\alpha_2 \mu_n \alpha} \int_0^t \sin \mu_n \alpha (t - \xi) f_1(\xi) \, d\xi + $$

$$+ U_0^p(n) \cos \mu_n \alpha t + \frac{U_0'(n)}{\mu_n \alpha} \sin \mu_n \alpha t \tag{19}$$

Finally applying the inversion (9), we obtain the required result as

$$u(r, t) = \sum_n \frac{1}{C_n} \left[\frac{b^2 M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b)}{\beta_2 \mu_n \alpha} \int_0^t \sin \mu_n \alpha (t - \xi) f_2(\xi) \, d\xi - \right.$$

$$- \frac{a^2 M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a)}{\alpha_2 \mu_n \alpha} \int_0^t \sin \mu_n \alpha (t - \xi) f_1(\xi) \, d\xi + $$

$$\left. + U_0^p(n) \cos \mu_n \alpha t + \frac{U_0'(n)}{\mu_n \alpha} \sin \mu_n \alpha t \right] M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) \tag{20}$$

CONVERGENCE OF THE INFINITE SERIES

Let us discuss the convergence of the infinite series (20) and investigate the conditions to be imposed on the functions $f_1(t)$, $f_2(t)$, $u_0(r)$ and $u_0'(r)$, so that the convergence of the series expansion for $u(r, t)$ in (20) is valid.

Considering the asymptotic behaviours of μ_n, C_n and $M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r)$ given in (12), (15) and (13) respectively and then on comparison with the auxiliary series $\sum_i \frac{1}{i^j}, j > 1$ and with[4] $a_n = \frac{\sin}{\cos} n\theta$ and $X_n = \frac{1}{n}$, we see that the series expansion (20) for $u(r, t)$ will be convergent if

$$(i) \int_0^t \sin \mu_n \alpha (t - \xi) \begin{matrix} f_1(\xi) \\ f_2(\xi) \end{matrix} \, d\xi = \left(\frac{1}{\mu_n^k} \right), \quad k > 0 ,$$

or

$$\int_0^t \sin \mu_n \alpha (t - \xi) \begin{matrix} f_1(\xi) \\ f_2(\xi) \end{matrix} \, d\xi = \left(\begin{matrix} \sin \\ \cos \end{matrix} n\theta \right)$$

$$(ii) \; U_0^p(n) = O(1) \quad (iii) \; U_0'^p(n) = O(n)$$

Example-1

$f_1(t)$ or $f_2(t)$ can be chosen as a finite sum or product of the following functions :

Constant, $\sin wt$, $\cos wt$, $\bar{e}^{kt}$, $\sum_{m=0}^{N} \delta(t - mt_0)$ and polynomials in t etc.

Example-2

$u_0(r)$ *or* $u_0'(r)$ can be chosen as a finite sum or product of the following functions :

Constant, $\sin wr$, $\cos wr$, $\bar{e}^{kr}$, $\sum_{m=0}^{N} (r - mr_0)$ or polynomials in r etc.

ILLUSTRATIONS

We now give some important practical illustrations of the general result (20).

Case-1

Let the internal and external pressures be periodic functions of time t with periods $2\pi/w_1$ and $2\pi/w_2$ respectively with initial conditions $u_0(r) = 0$ and $u_0'(r) = 0$, then

$f_1(t) = -A_1(1 - \cos w_1 t)$, $f_2(t) = -A_2(1 - \cos w_2 t)$, $u_0(r) = 0$ and $u_0'(r) = 0$.

Substituting the above boundary and initial conditions in the general solution (20), and simplifying, we get

$$
\begin{aligned}
u(r,t) = \sum_n \frac{1}{C_n} \Bigg\{ & \frac{b^2 A_2}{\beta_2 \mu_n \alpha} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b) \Bigg[\frac{\cos \mu_n \alpha t - 1}{\mu_n \alpha} + \\
& + \tfrac{1}{2} \sin \mu_n \alpha t \left(\frac{\sin(w_2 + \mu_n \alpha) t}{w_2 + \mu_n \alpha} + \frac{\sin(w_2 - \mu_n \alpha) t}{w_2 - \mu_n \alpha} \right) - \\
& - \tfrac{1}{2} \cos \mu_n \alpha t \left(\frac{1 - \cos(w_2 + \mu_n \alpha) t}{w_2 + \mu_n \alpha} + \frac{1 - \cos(w_2 - \mu_n \alpha) t}{w_2 - \mu_n \alpha} \right) \Bigg] - \\
& - \frac{a^2 A_1}{\alpha_2 \mu_n \alpha} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a) \Bigg[\frac{\cos \mu_n \alpha t - 1}{\mu_n \alpha} + \\
& + \tfrac{1}{2} \sin \mu_n \alpha t \left(\frac{\sin(w_1 + \mu_n \alpha) t}{w_1 + \mu_n \alpha} + \frac{\sin(w_1 - \mu_n \alpha) t}{w_1 - \mu_n \alpha} \right) - \\
& - \tfrac{1}{2} \cos \mu_n \alpha t \left(\frac{1 - \cos(w_1 + \mu_n \alpha) t}{w_1 + \mu_n \alpha} + \frac{1 - \cos(w_1 - \mu_n \alpha) t}{w_1 - \mu_n \alpha} \right) \Bigg] \Bigg\} \times \\
& \times M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r).
\end{aligned} \tag{21}
$$

By putting $A_1 = A$ and $A_2 = 0$ in the above result, we get the solution of the problem solved by Sneddon[1].

Case-2

Let the internal and external pressures be exponentially decreasing with the initial conditions $u_0(r) = 0$ and $u_0'(r) = 0$,

then

$$f_1(t) = -A_1 e^{-k_1 t}, \quad f_2(t) = -A_2 e^{-k_2 t}, \quad u_0(r) = 0, \; u_0'(r) = 0.$$

Substituting the above boundary and initial conditions in the general solution (20), we obtain

$$
\begin{aligned}
u(r,t) = \sum_n \frac{1}{C_n} \Bigg[& \frac{a^2 A_1}{\alpha_2 \mu_n \alpha} \frac{M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a)}{K_1^2 + \mu_n^2 \alpha^2} (e^{-k_1 t} + k_1 \sin \mu_n \alpha t - \\
& - \mu_n \alpha \cos \mu_n \alpha t) - \frac{b^2 A_2}{\beta_2 \mu_n \alpha} \frac{M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, b)}{K_2^2 + \mu_n^2 \alpha^2} \times \\
& \times (e^{-k_2 t} + k_2 \sin \mu_n \alpha t - \mu_n \alpha \cos \mu_n \alpha t) \Bigg] M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r).
\end{aligned} \tag{22}
$$

Case-3

Let the internal pressure be a periodic succession of impulses with a period t_0 and the external pressure be zero with initial conditions $u'_o(r) = 0$ and $u'_o(r) = 0$,

then

$$f_1(t) = -A \sum_{m=0}^{N} \delta(t - mt_0),\ f_2(t) = 0,\ u_0(r) = 0,\ u_0'(r) = 0.$$

Substituting the above boundary and initial conditions in the general solution (20), we obtain

$$u(r,t) = \sum_{0<m<\frac{t}{t_0}} \sum_{n} \frac{1}{C_n} \frac{\alpha^2 A}{\alpha_2 \mu_n \alpha} M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, a)$$

$$\times \operatorname{Sin} \mu_n \alpha (t - mt_0) M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_n, r) \tag{23}$$

NUMERICAL CALCULATIONS

Let us consider a hollow sphere of radii 0·5 m and 1·0 m. Let the material of the sphere be rolled copper for which the elastic constants are as given below[5]

$$\lambda = 13{\cdot}1 \times 10^{10} \text{ Newtons}/m^2,\ \eta = 4{\cdot}6 \times 10^{10} \text{ Newtons}/m^2,\ \alpha = \sqrt{\frac{\lambda + 2\eta}{\rho}} = 5010\ m/sec.,$$

so that

$$\alpha_2 = \beta_2 = \lambda + 2\eta = 22{\cdot}3 \times 10^{10},\ \alpha_1 = \frac{2\lambda}{a} = 52{\cdot}4 \times 10^{10}$$

and

$$\beta_1 = \frac{2\lambda}{b} = 26{\cdot}2 \times 10^{10}.$$

Following the procedure of Mclachlan[3], we obtain the first positive root of (7) as $\mu_1 = 1{\cdot}799$ for $a = 0{\cdot}5$m. and $b = 1{\cdot}0$ m and then $C_1 = 93725 \times 10^{20}$ and values of $M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, r)$ for different values of r are given below :—

r	0·5	0·6	0·7	0·8	0·9	1·0
$M_p(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, r)$	$124{\cdot}7\times10^{10}$	$61{\cdot}67\times10^{10}$	$72{\cdot}81\times10^{10}$	$73{\cdot}18\times10^{10}$	$124{\cdot}6\times10^{10}$	$202{\cdot}9\times10^{10}$

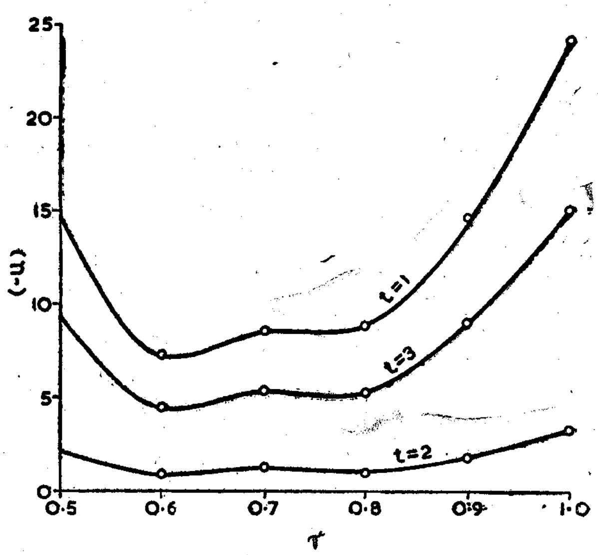

Fig. 1— $(-u)$ Vs r for various values of t.

Case-1

Taking $A_1 = A_2 = 10^{21}$, $w_1 = \pi/2$, $w_2 = \pi$ in (21), we obtain the variations of $-u$ with r as shown in figure 1, for $t = 1$, 2 or 3 seconds.

Case-2

Taking $A_1 = A_2 = 10^{21}$, $k_1 = 0{\cdot}01$, $k_2 = 0{\cdot}05$ in (22), we obtain the variations of $-u$ with r as shown in figure 2, for $t = 1$, 2 or 3 seconds.

Case-3

Taking $A = 10^{20}$, $t_0 = 1$ sec. in (23), we obtain the variations of u with r as shown in figure 3, for $t = 1/2$, $3/2$ or $5/2$ seconds.

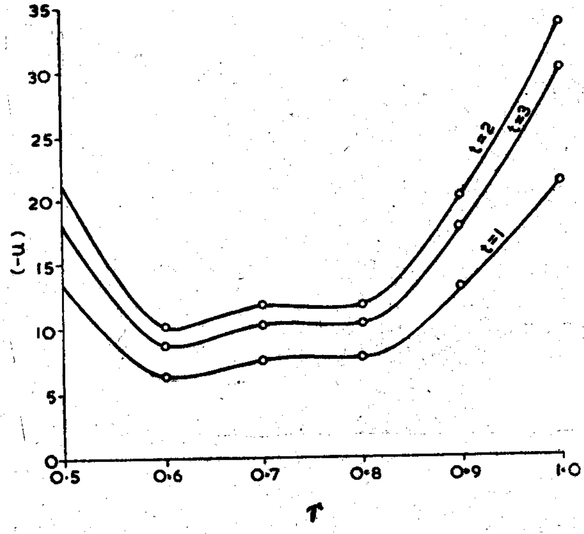

Fig. 2— (—u) Vs r for various values of t.

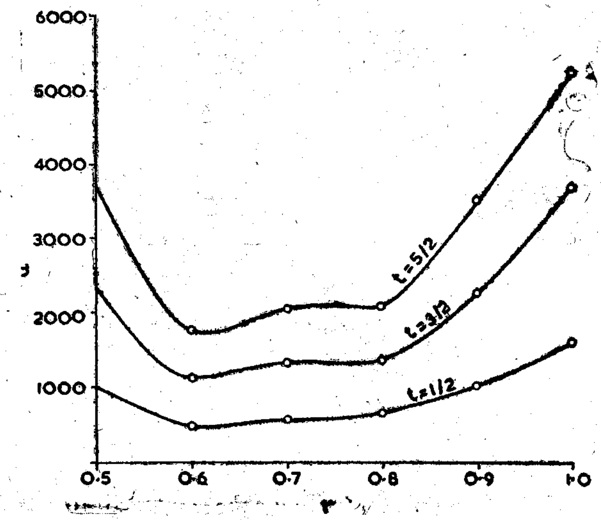

Fig. 3— (u) Vs r for various values of t.

CONCLUSION

This paper deals with a general problem of vibrations produced in a thick hollow sphere. Any particular case of special interest can be obtained by assigning suitable values to the parameters and functions involved in the general solution (20).

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