THREE DIMENSIONAL SHOCK WAVES IN A GASEOUS MEDIUM CONTAINING DUST PARTICLES

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In this paper the discontinuous jump in the flow parameters such as density, pressure, energy, velocity, enthalpy, entropy, etc., across the three dimensional shock wave in a dusty medium which is a homogeneous mixture of a perfect gas and dust particles with no heat conduction or viscosity has been computed. We have also obtained the relations for density shock strength, volume shock strength and pressure shock strength respectively in a dusty medium.

So far most of the studies on shock waves have been done on the assumption that the underlying medium is a perfect gas with no heat conduction or viscosity. But, in practice, all real mediums exhibit the phenomena of conduction and viscosity. Also there may be traces of small solid particles in the medium. The purpose of this paper is to compute the discontinuous jump in density, pressure, velocity, enthalpy, etc., across a three dimensional shock wave in a dusty medium, which is a homogeneous mixture of a perfect gas and dust particles with no heat conduction and viscosity. The dust particles are assumed to be incompressible and evenly distributed in the medium.

Carrier[1] was first to study shock waves in a dusty medium, Mishra and Srivastava[2,3] studied some properties of the shocks in a dusty medium and showed that the presence of dust particles in a medium reduces the density as well as the velocity shock strength.

FUNDAMENTAL EQUATIONS

Following Campbell and Pitcher[4] let $\rho^* = \rho^*(x^i, t)$ be the density of the dusty medium at a point $P(x^i)$ at time t and $\rho_g = \rho_g(x^i, t)$ be the density of the perfect gas. Then if $\rho_0 = \rho_0(x^i, t)$ is the density of the individual dust particle and μ be the ratio of the mass of the gas to the mass of the dust, then the mean density of the mixture ρ^* is given by the relation

$$\frac{1+\mu}{\rho^*} = \frac{\mu}{\rho_g} + \frac{1}{\rho_0}$$

or

$$\rho^* = \frac{(1+\mu)\,\rho_0\,\rho_g}{(\mu\rho_0 + \rho_g)} \quad (1)$$

The Rankine-Hugôniots jump conditions in dusty medium can be written as

$$\frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0 + \rho_{1g})}\,u_{1n} = \frac{(1+\mu)\,\rho_0\,\rho_{2g}}{(\mu\rho_0 + \rho_{2g})}\,u_{2n} = m_g + m_d, \quad (2)$$

$$\Big[\,p\,\Big] + (m_g + m_d)\Big[\,u_n\,\Big] = 0, \quad (3)$$

$$\Big[\,u_\alpha\,\Big] = 0 \quad (4)$$

and

$$\tfrac{1}{2}(m_g + m_d)\Big[\,u^2\,\Big] + m_g\Big[\,E\,\Big] + m_d\Big[\,\epsilon\,\Big] + \Big[\,p u_n\,\Big] = 0,$$

where $v_a \overset{def}{=} u_i\, x^i_{,\alpha}\ (\alpha = I, II)$, $x^i_{,\alpha}$ being tangential to shock surface and the square brackets [ ] denote the difference of values on the two sides of the shock surface, of the quantity enclosed.

Thus, if f be any parameter, then $[f]=f_2-f_1$. Also, if T is the temperature and C_g and C_d are the specific heats for the gas and the dust respectively, then

$$E \overset{def}{=} C_g T$$

and
$$\epsilon = C_{\alpha} T \tag{6}$$

Definition 1 : The dimensionless quantity δ^*_ρ defined by

$$\delta^*_\rho = \frac{[\rho^*]}{\rho^*_1} \tag{7}$$

is called the density shock strength of the dusty medium. Similarly, volume shock strength δ^*_τ and pressure shock strength δ^*_p are given by

$$\delta^*_\tau = \frac{[\tau^*]}{\tau_1^*} = -\frac{\delta^*_\rho}{1+\delta^*_\rho} \tag{8}$$

$$\delta^*_p = \frac{[p]}{p_1}, \tag{9}$$

where $\tau^* = \dfrac{1}{\rho^*}$ is the specific volume. Density shock strength of the gaseous medium δ_{ρ_g} is defined by

$$\delta_{\rho_g} = \frac{[\rho_g]}{\rho_{1g}} = \frac{\rho_{2g}-\rho_{1g}}{\rho_{1g}}, \tag{10}$$

where ρ_{1g} denotes the density of the gas in front of the shock and ρ_{2g} denotes the density of the gas behind the shock.

Theorem 1 : In terms of δ^*_ρ the Rankine-Hugoniot equations (2)—(4) are equivalent to :

$$\delta^*_\rho = \frac{\mu\rho_0\,\delta_{\rho_g}}{\mu\rho_0 + \rho_{1g}\,(1+\delta_{\rho_g})}, \tag{11}$$

$$[u^i] = -\frac{\mu\rho_0\,\delta_{\rho_g}}{(\mu\rho_0+\rho_{1g})}\cdot\frac{1}{(1+\delta_{\rho_g})}\,u_{1n},\ X^i \tag{12}$$

and
$$[p] = \frac{\mu\,(1+u)\,\rho_0^2\,\rho_{1g}\,\delta_{\rho_g}}{(\mu\rho_0+\rho_{1g})^2\,(1+\delta_{\rho_g})}\,u_{1n}^2 \tag{13}$$

Proof : From equations (2)—(4), we have

$$[p] = \rho_1^*\,u^2_{1n}\,\frac{[\rho^*]}{\rho^*_2} = \frac{\mu\,(1+\mu)\,\rho_0^2\,\rho_{1g},\,\delta_{\rho_g}}{(\mu\rho_0+\rho_{1g})^2\,(1+\delta_{\rho_g})}\cdot u_{1n}^2$$

The vector u^i can be expressed as a linear combination of three non-coplanar vectors. Thus, $u^i = u_n X^i + u^\alpha x^i{}_{,\alpha}$. From this equation we get

$$[u^i] = [u_n]\,X^i + [u^a]\,x^i{}_{,\alpha} \tag{14}$$

From (2) and (7), we have

$$[u_n] = -\frac{[\rho^*]}{\rho^*_2}\,u_{1n} \quad \text{or}$$

$$[u_n] = -\frac{\delta^*_\rho}{1+\delta^*_\rho}\,u_{1n} = -\frac{\mu\rho_0\,\delta_{\rho_g}}{(\mu\rho_0+\rho_{1g})}\cdot\frac{1}{(1+\delta_{\rho_g})}\,u_{1n} \tag{15}$$

Also, if $g_{\alpha\beta} \overset{def}{=} x^i{}_{,\alpha}\, x^i{}_{,\beta}$ and $g_{\alpha\beta}\, g^{\alpha\gamma} = \delta_\beta{}^\gamma$†, then

$$[\, u^\alpha \,] = [\, u_\beta\, g^\alpha{}_\beta \,] = g^\alpha{}_\beta\, [\, u_\beta \,] = 0 \tag{16}$$

in consequence of (4). Substituting from (15) and (16) in (14), we obtain (12).

Corollary 1 : In terms of volume strength and pressure strength, the equations (11), (12), (13), assume the forms

$$\delta^*{}_\tau = \frac{[\, \tau^* \,]}{\tau^*{}_1} = - \frac{\delta^*{}_\rho}{1+\delta^*{}_\rho} = -\left(\frac{\mu\rho_0\, \delta_{\rho_g}}{\mu\rho_0 + \rho_{1g}}\right) \cdot \frac{1}{1+\delta_{\rho_g}}\,, \tag{17}$$

$$[\, u^i \,] = \delta^*{}_\tau\, u_{1n}\, X^i, \tag{18}$$

$$[\, p \,] = - \frac{\delta^*{}_\tau}{\tau^*{}_1}\, u^2{}_{1n}\,, \tag{19}$$

$$\delta^*{}_p = \frac{[\, p \,]}{p_1}\,, \tag{20}$$

$$[\, u^i \,] = - \frac{\delta^*{}_p}{\left(\dfrac{(1+\mu)\ \rho_0\, \rho_{1g}}{\mu\rho_0 + \rho_{1g}}\right) \cdot \dfrac{u_{1n}}{p_1}}\, X^i \tag{21}$$

and

$$[\, \rho^* \,] = \frac{(1+\mu)\, p^2{}_1\, \delta^*{}_p}{(1+\mu)\, u^2{}_{1n} - p_1{}^2\, \delta^*{}_p \left(\dfrac{\mu}{\rho_{1g}} + \dfrac{1}{\rho_0}\right)} \tag{22}$$

The proof follows by simple computation.

We know that the quantity h defined by

$$h \overset{def}{=} H + \frac{p}{\dfrac{(1+\mu)\ \rho_0\, \rho_g}{(\mu\rho_0 + \rho_g)}}\,, \tag{23}$$

where H the internal energy per unit mass, is the enthalpy. In the case of dusty medium H is given by

$$H = \frac{m_g\, C_g + m_d\, C_d}{m_g + m_d}\, T\,, \quad \text{or} \tag{24}$$

$$H = \frac{\phi}{\gamma - 1} \cdot \frac{p}{\dfrac{(1+\mu)\ \rho_0\, \rho_g}{(\mu\rho_0 + \rho_g)}}\,, \tag{25}$$

where $p = \dfrac{(1+\mu)\ \rho_0\, \rho_g}{(\mu\rho_0 + \rho_g)}\, RT$, $R = C_p - C_g$ and ϕ is dimensionless quantity defined by

$$\phi \overset{def}{=} \frac{m_g + m_d\, \dfrac{C_d}{C_g}}{m_g + m_d}\,. \tag{26}$$

Theorem 2 : The jump in enthalpy across the shock surface is given by

$$[\, h \,] = \tfrac{1}{2} \cdot \frac{\mu\, \rho_0\, \delta_{\rho_g}}{(\mu\rho_0 + \rho_{1g})^2\, (1+\delta_{\rho_g})^2} \left\{ \mu\rho_0\, (2 + \delta_{\rho_g}) + 2\rho_g\, (1+\delta_{\rho_g}) \right\} u_{1n}{}^2 \tag{27}$$

†It is possible to define $g^{\alpha\beta}$ in this way, because the shock surface being a real surface $Det\, |\, g_{\alpha\beta}\, | \neq 0$.

Proof : From (18), we obtain

$$[u^i] = - \frac{\mu\, \rho_0\, \delta_{\rho_g}}{(\mu\, \rho_0 + \rho_{1g})^2} \cdot \frac{1}{(1 + \delta_{\rho_g})} \cdot u_{1n}\, X^i$$

From (15), we obtain

$$[u^2] = - \frac{\mu\, \rho_0\, \delta_{\rho_g}}{(\mu \rho_0 + \rho_{1g})^2\, (1 + \delta_{\rho g})^2} \left\{ \mu\, \rho_0\, (2 + \delta_{\rho_g}) + 2\rho_{1g}\, (1 + \delta_{\rho_g}) \right\} u^2_{1n} \tag{28}$$

Now

$$\left[\frac{p}{\frac{(1 + \mu)\, \rho_0\, \rho_g}{(\mu \rho_0 + \rho_g)}} \right] = \frac{p_2}{\frac{(1 + \mu)\, \rho_0\, \rho_{2g}}{(\mu \rho_0 + \rho_{2g})}} - \frac{p_1}{\frac{(1 + \mu)\, \rho_0\, \rho_{1g}}{(\mu \rho_0 + \rho_{1g})}}$$

Also in consequence of (2) and (6), the equation (5) assumes the form

$$\tfrac{1}{2}\, (m_g + m_d)\, [u^2] + (m_g\, C_g + m_d\, C_d)\, [T] + (m_g + m_d) \cdot \left[\frac{p}{\frac{(1 + \mu)\, \rho_0\, \rho_g}{(\mu \rho_0 + \rho_g)}} \right] = 0$$

Using (28) and (24) in this equation, we obtain

$$\tfrac{1}{2}\, [u^2] + \left[H + \frac{p}{\frac{(1+\mu)\, \rho_0\, \rho_g}{(\mu \rho_0 + \rho_g)}} \right] = 0 \tag{29}$$

In consequence of (23), this equation assumes the form (27).

The quantity S defined by

$$S \overset{def}{=} J\, C_g \log \frac{p}{\left(\frac{(1+\mu)\, \rho_0\, \rho_g}{\mu \rho_0 + \rho_g} \right)^{\gamma}} \tag{30}$$

is called the entropy of the gas.

Theorem 3 : The jump in entropy across the shock surface is given by

$$[S] = J\, C_g \log \left\{ \left(1 + \frac{\mu\, (1 + \mu)\, \rho_0^2\, \rho_{1g}\, \delta_{\rho_g}}{(\mu \rho_0 + \rho_{1g})^2\, (1 + \delta_{\rho g})} \cdot \frac{u^2_{1n}}{p_1} \right) \left((1 + \delta_{\rho_g}) \cdot \frac{\mu \rho_0 + \rho_{1g}}{\mu \rho_0 + \rho_{1g}\, (1 + \delta_{\rho_g})} \right)^{-\gamma} \right\} \tag{31}$$

Proof : From (30), we have

$$[S] = J\, C_g \log \left\{ \frac{p_2}{p_1} \cdot \left(\frac{\mu \rho_0 + \rho_{1g}}{\mu \rho_0 + \rho_{1g}\, (1 + \delta_{\rho_g})} \cdot (1 + \delta_{\rho_g}) \right)^{-\gamma} \right\}$$

Now

$$\frac{p_2}{p_1} = 1 + \frac{[p]}{p_1}$$

Substituting from (13) in this equation, we obtain (31).

It is well known that the quantity C defined by

$$C^2 \overset{def}{=} \left(\frac{\partial p}{\partial \rho^*} \right)_s \tag{32}$$

is the velocity of sound. In case of perfect dusty medium, it is given by

$$C^2 = \frac{\gamma p_1}{\frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0+\rho_{1g})}} = \frac{(\mu\rho_0+\rho_{1g})}{(1+\mu)\,\rho_0\,\rho_{1g}}\cdot\gamma p_1 \tag{33}$$

Note : It may be noted that δ^* and $M_{\alpha n}$ are dimensionless quantities. $M_{\alpha n}$ is called the Mach number of the velocity normal to the shock surface in the α-th region.

CONSEQUENCES OF RANKINE-HUGONIOT EQUATIONS

The equation (4) is a statement of the fact that the tangential components of velocity are continuous across the shock surface even in dusty medium.

Theorem 4 : We have

$$[p] = -(1+\mu)\,\rho_0\left[\frac{\rho_g}{\mu\rho_0+\rho_g}\cdot u^2_{1n}\right], \tag{34}$$

$$u_{2n} = \frac{\mu\rho_0+\rho_{1g}(1+\delta_{\rho_g})}{(\mu\rho_0+\rho_{1g})(1+\delta_{\rho_g})}\cdot u_{1n}, \tag{35}$$

$$[h] = -\tfrac{1}{2}\left[u^2_n\right] \tag{36}$$

Proof : Substituting from (1) and (2) in (3), we obtain (34). Multiplying (12) by X^i and summing for i, we obtain (35). Combining equations (35) and (27), we obtain (36).

Definition 2 : The quantity whose components $\phi_{r/\alpha}$ defined by

$$\phi_{r/\alpha} \overset{def}{=} \frac{u_\alpha}{u_{rn}} \qquad r = 1, 2 \tag{37}$$

is called the obliquity of the shock surface in the r-th region.

Theorem 5 : We have

$$[\phi_{r/\alpha}] = \frac{\mu\rho_0\,\delta_{\rho_g}}{\mu\rho_0+\rho_{1g}(1+\delta_{\rho_g})}\cdot\phi_{1/\alpha} \tag{38}$$

equivalent to

$$\phi_{2/\alpha} = \frac{(\mu\rho_0+\rho_{1g})(1+\delta_{\rho_g})}{\mu\rho_0+\rho_{1g}(1+\delta_{\rho_g})}\cdot\phi_{1/\alpha}. \tag{39}$$

Consequently, the obliquity is proportional to density.

Proof : Substituting from (35) in (37), we obtain (38). From (35), we obtain (39).

Theorem 6 : The tangents to the stream lines on opposite sides of the shock surface and normal to the shock surface at a point P are coplaner.

Proof : A linear relation (12) exists between $u^i{}_{,\alpha}$ and X^i. Hence, we have the theorem.

VOLUME SHOCK STRENGTH

Theorem 7 : The Rankine-Hugoniot equations are

$$[u^i_1] = \delta_\tau^*\,u_{1n}\,X^i \tag{40}$$

$$[p] = -\frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0+\rho_{1g})}\,\delta_\tau^*\,u^2_{1n}, \tag{41}$$

$$[\rho_1^*] = -\frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0+\rho_{1g})}\cdot\frac{\delta_\tau^*}{(1+\delta_\tau^*)} \tag{42}$$

Proof : From (14) and (17), we get (40). From (2), (3) and (7), we get (41). From (7) and (8), we get (42).

Theorem 8 : The jump in entropy is given by

$$[S] = J\, C_g \log \left\{ \left(1 - \frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0 + \rho_{1g})\,p_1} \cdot \delta_\tau^* \, u^2_{1n} \right) (1+\delta_\tau^*)^\gamma \right\} \tag{43}$$

Proof : Using (41) in $[S] = J\, C_g \left[\log \dfrac{p}{\left(\dfrac{(1+\mu)\,\rho_0\,\rho_g}{\mu\,\rho_0 + \rho_g} \right)^\gamma} \right]$, we get (43).

PRESSURE SHOCK STRENGTH

Following Mishra and Srivastava[2] we have the Rankine-Hugoniot jump conditions as

$$[u^i] = - \frac{(\mu\,\rho_0 + \rho_{1g})}{(1+\mu)\,\rho_0\,\rho_{1g}} \cdot \frac{p_1 \delta_p^*}{u_{1n}} X^i, \tag{44}$$

$$[p] = \delta_p^* \, p_1 \, , \tag{45}$$

$$[\rho^*] = \frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0 + \rho_{1g})} \cdot \frac{1}{\dfrac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\,\rho_0 + \rho_{1g})} \cdot \dfrac{u^2_{1n}}{p_1} - \delta_p^*} \cdot \delta_p^* \tag{46}$$

Theorem 9 : The jump in entropy is given by

$$[S] = J\, C_g \log \left\{ (1+\delta_p^*) \left(1 - \frac{(\mu\,\rho_0 + \rho_{1g})}{(1+\mu)\,\rho_0\,\rho_{1g}} \cdot \frac{\delta_p^* \, p_1}{u^2_{1n}} \right)^\gamma \right\} \tag{47}$$

Proof : Using $[S] = J\, C_g \left[\log p \left\{ \dfrac{(1+\mu)\rho_0\,\rho_g}{(\mu\,\rho_0 + \rho_g)} \right\}^{-\gamma} \right]$ and (45), (46), we have (47).

Theorem 10 : The jump in entropy can also be written as

$$[S] = J\, C_g \log \left\{ (1+\delta_p^*)\,(1+\delta_\tau^*)^\gamma \right\} \tag{48}$$

Proof : Using (42) and (43), we get (48).

Theorem 11 : The jump in enthalpy is given by

$$[h] = \frac{\delta_p^*}{\left(\dfrac{(1+\mu)\,\rho_0\,\rho_{1g}}{\mu\rho_0 + \rho_{1g}} \right)^2 \cdot \dfrac{u^2_{1n}}{p_1^2}} \left\{ \frac{(1+\mu)\,\rho_0\,\rho_{1g}}{(\mu\rho_0 + \rho_{1g})} \cdot \frac{u_{1n}^2}{p_1} - \tfrac{1}{2}\,\delta_p^* \right\} \tag{49}$$

Proof : Using the square of (44) in (5), (2), (23), we get (49).

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